

# Tangent unit-vector fields: nonabelian homotopy invariants and the Dirichlet energy

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## Abstract

Let  $O$  be a closed geodesic polygon in  $S^2$ . Maps from  $O$  into  $S^2$  are said to satisfy tangent boundary conditions if the edges of  $O$  are mapped into the geodesics which contain them. Taking  $O$  to be an octant of  $S^2$ , we compute the infimum Dirichlet energy,  $\mathcal{E}(H)$ , for continuous maps satisfying tangent boundary conditions of arbitrary homotopy type  $H$ . The expression for  $\mathcal{E}(H)$  involves a topological invariant – the spelling length – associated with the (nonabelian) fundamental group of the  $n$ -times punctured two-sphere,  $\pi_1(S^2 - \{s_1, \dots, s_n\}, *)$ . The lower bound for  $\mathcal{E}(H)$  is obtained from combinatorial group theory arguments, while the upper bound is obtained by constructing explicit representatives which, on all but an arbitrarily small subset of  $O$ , are alternatively locally conformal or anticonformal. For conformal and anticonformal classes (classes containing wholly conformal and anticonformal representatives respectively), the expression for  $\mathcal{E}(H)$  reduces to a previous result involving the degrees of a set of regular values  $s_1, \dots, s_n$  in the target  $S^2$  space. These degrees may be viewed as invariants associated with the abelianization of  $\pi_1(S^2 - \{s_1, \dots, s_n\}, *)$ . For nonconformal classes, however,  $\mathcal{E}(H)$  may be strictly greater than the abelian bound. This stems from the fact that, for nonconformal maps, the number of preimages of certain regular values may necessarily be strictly greater than the absolute value of their degrees.

This work is motivated by the theoretical modelling of nematic liquid crystals in confined polyhedral geometries. The results imply new lower and upper bounds for the Dirichlet energy (one-constant Oseen-Frank energy) of reflection-symmetric tangent unit-vector fields in a rectangular prism.

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# 1 Statement of Results

Let  $\mathcal{C}(S^2, S^2)$  denote the space of continuous maps of the two-sphere into itself. As is well known, maps in  $\mathcal{C}(S^2, S^2)$  are classified up to homotopy by their degree, and the infimum of the Dirichlet energy,  $\int_{S^2} |\phi'|^2 dA$ , for maps  $\phi$  of given degree  $d$  is equal to  $8\pi d$  (the area element  $dA$  is normalised so that  $S^2$  has area  $4\pi$ ). Critical points of the Dirichlet energy include conformal or anticonformal maps, and the infimum energy may be realised by conformal maps (for  $d \geq 0$ ) and anticonformal maps (for  $d \leq 0$ ) on  $S^2$ .

In this paper we study an elaboration of this problem, motivated, as explained in Section 1.1 below, by models of nematic liquid crystals in confined polyhedral geometries. Consider a set of  $f$  geodesics (great circles) on  $S^2$ . The geodesics divide  $S^2$  into a collection of closed spherical polygons (Euler's theorem implies that, generically, there are  $f^2 - f + 2$  polygons). Let  $O$  denote one such polygon. A map  $\nu : O \rightarrow S^2$  is said to satisfy *tangent boundary conditions* if  $\nu$  maps each edge of  $O$  into the geodesic which contains it. Let  $\mathcal{C}_T(O, S^2)$  denote the set of continuous maps  $\nu : O \rightarrow S^2$  which satisfy tangent boundary conditions. We may then ask, for a given homotopy class  $H$  in  $\mathcal{C}_T(O, S^2)$ , what is the infimum of the Dirichlet energy?

In this paper we will take  $O$  to be the positive coordinate octant,

$$O = \{r \in S^2 \subset \mathbb{R}^3 \mid r_j \geq 0\}, \quad (1)$$

whose edges lie on the three coordinate geodesics, ie the unit circles about the origin in the  $xy$ -,  $yz$ - and  $zx$ -planes. The vertices of  $O$  are the coordinate unit vectors  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{z}}$ .

The homotopy classification of  $\mathcal{C}_T(O, S^2)$  is described in [18, 12, 14]. Let us summarize the relevant results. First, given  $\nu \in \mathcal{C}_T(O, S^2)$ , consider the values of  $\nu$  at the vertices of  $O$ . Tangent boundary conditions imply that  $\nu(\hat{j}) = e_j \hat{j}$ , where  $e_j = \pm 1$ . The  $e_j$ 's, called *edge signs*, are homotopy invariants. Consider next the values of  $\nu$  on the edges of  $O$ . The image of the  $yz$ -edge (for example) under  $\nu$  is a curve on the  $yz$ -coordinate circle with endpoints  $e_y \hat{\mathbf{y}}$  and  $e_z \hat{\mathbf{z}}$ . The integer-valued winding number of this curve relative to the shortest geodesic between its endpoints is another invariant, called the *kink number*, which we denote by  $k_x$ . (To be consistent with our previous conventions, we take  $k_x \leq 0$  if  $\nu$  preserves orientation on the  $yz$ -edge.) The kink numbers  $k_y$  and  $k_z$  are defined similarly. Finally, the oriented area of the image of the interior of  $O$  under  $\nu$ , denoted  $\Omega$ , is also an invariant, called the *trapped area*. For  $\nu$  differentiable, the trapped area is given by

$$\Omega = - \int_O \nu^* \omega, \quad (2)$$

where  $\omega$  is the area two-form on  $S^2$  ( $\omega$  is normalised so that  $\int_{S^2} \omega = 4\pi$ ). (To be consistent with our previous conventions, we have taken  $\Omega < 0$  for  $\nu$  orientation-preserving.) For a given set of edge signs  $e = (e_x, e_y, e_z)$  and kink numbers  $k = (k_x, k_y, k_z)$ , the allowed values of  $\Omega$  differ by integer multiples of  $4\pi$ . The invariants  $(e, k, \Omega)$  collectively classify the homotopy classes of  $\mathcal{C}_T(O, S^2)$ , and all allowed values can be realised.

A second classification scheme, described in [12, 14], is based on generalised degrees. Observe that the coordinate geodesics which define the domain  $O$  also partition the target  $S^2$  space into open

coordinate octants, here called *sectors*. We label the sectors by fixing the signs of the coordinates. Thus, we let  $\sigma = (\sigma_x \sigma_y \sigma_z)$  denote a triple of signs, and define the sector  $\Sigma_\sigma$  by

$$\Sigma_\sigma = \{s \in S^2 \subset \mathbb{R}^3 \mid \sigma_j s_j > 0\}. \quad (3)$$

(Thus,  $O$  is the closure of  $\Sigma_{+++}$ , although we shall regard  $O$  and  $cl(\Sigma_{+++})$  as distinct, with  $O$  constituting the domain of  $\nu$  and  $cl(\Sigma_{+++})$  constituting a subset of the target space.) It can be shown that the degree of a regular value  $s_\sigma \in \Sigma_\sigma$  of  $\nu$ , ie, the number of preimages of  $s_\sigma$  counted with a sign according to orientation, is a homotopy invariant called the *wrapping number*, which we denote by  $w_\sigma$ . For  $\nu$  differentiable,

$$w_\sigma = - \sum_{x \in \nu^{-1}(s_\sigma)} \text{sgn det } \nu'(x) \quad (4)$$

(To be consistent with our previous conventions, we have taken  $w_\sigma \leq 0$  for  $\nu$  orientation-preserving.) Using Stokes' theorem, one can express the wrapping numbers in terms of  $(e, k, \Omega)$ ,

$$w_\sigma = \frac{1}{4\pi} \Omega + \frac{1}{2} \sum_j \sigma_j k_j + e_x e_y e_z \left( \frac{1}{8} - \delta_{\sigma, e} \right). \quad (5)$$

(5) can be inverted to obtain  $(e, k, \Omega)$  in terms of the  $w_\sigma$ 's. Thus the wrapping numbers  $w_\sigma$  are a (constrained) set of classifying invariants for  $\mathcal{C}_T(O, S^2)$ .

We say that a homotopy class in  $\mathcal{C}_T(O, S^2)$  is *conformal* if  $w_\sigma \leq 0$  for all  $\sigma$ , *anticonformal* if  $w_\sigma \geq 0$  for all  $\sigma$ , and *nonconformal* otherwise. In [12] it is shown that every conformal homotopy class has a conformal representative. In terms of the complex coordinate  $w = (s_x + i s_y)/(1 + s_z)$  on  $S^2$  (for  $\mathbf{s} = (s_x, s_y, s_z) \in S^2$ ), the conformal representatives are rational functions whose zeros and poles satisfy constraints dictated by tangent boundary conditions. Likewise, every anticonformal homotopy class has an anticonformal representative which is a rational function of  $\bar{w}$ . Representatives for nonconformal topologies are also discussed in [12].

We say that the sectors labeled by  $\sigma$  and  $\sigma'$  are *adjacent*, denoted  $\sigma \sim \sigma'$ , if  $\Sigma_\sigma$  and  $\Sigma_{\sigma'}$  share a common edge, or equivalently, if  $\sigma$  and  $\sigma'$  have precisely two components the same.

Given  $\nu \in \mathcal{C}_T(O, S^2) \cap W^{1,2}(O, S^2)$ , let

$$E(\nu) = \int_O |\nu'|^2 dA \quad (6)$$

denote the Dirichlet energy of  $\nu$ . Given a homotopy class  $H \subset \mathcal{C}_T(O, S^2)$ , let

$$\mathcal{E}(H) = \inf_{\nu \in H} E(\nu) \quad (7)$$

denote the infimum Dirichlet energy over  $H$ . Our main result, contained in Theorems 1 and 2 below, is an explicit formula for  $\mathcal{E}(H)$ . The formula consists of two contributions. The first,  $\sum_\sigma |w_\sigma| \pi$ , which on its own constitutes a lower bound for  $\mathcal{E}(H)$ , follows from considerations of the algebraic degree. This is analogous to what one has for maps in  $\mathcal{C}(S^2, S^2)$ .

The additional contribution involves a new homotopy invariant,  $\Delta(H)$ , which we now define. First, we have that

$$\Delta(H) = 0, \quad H \text{ conformal or anticonformal.} \quad (8)$$

For  $H$  nonconformal, let  $\sigma_+$  label a sector with the largest positive wrapping number (in cases where  $\sigma_+$  is not unique, the definition (10) below does not depend on the choice of  $\sigma_+$ .) Similarly, let  $\sigma_-$  denote the sector with the smallest negative wrapping number (ie, negative wrapping number of largest magnitude). Let

$$\chi = \begin{cases} 1, & k_x k_y k_z < 0, \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

Then we define

$$\Delta(H) = 2 \max \left( 0, w_{\sigma_+} - \sum_{\sigma \sim \sigma_+} \Phi(w_\sigma) - \chi, |w_{\sigma_-}| - \sum_{\sigma \sim \sigma_-} \Phi(-w_\sigma) - \chi \right), \quad H \text{ nonconformal,} \quad (10)$$

where

$$\Phi(x) = \frac{1}{2}(x + |x|). \quad (11)$$

We now state our main results.

**Theorem 1.** *Let  $H$  be a homotopy class in  $\mathcal{C}_T(O, S^2)$ . Then*

$$\mathcal{E}(H) \geq \left( \sum_{\sigma} |w_\sigma| + \Delta(H) \right) \pi. \quad (12)$$

**Theorem 2.** *Let  $H$  be a homotopy class in  $\mathcal{C}_T(O, S^2)$ . Then*

$$\mathcal{E}(H) \leq \left( \sum_{\sigma} |w_\sigma| + \Delta(H) \right) \pi. \quad (13)$$

**Corollary 1.** *Let  $H$  be a homotopy class in  $\mathcal{C}_T(O, S^2)$ . Then*

$$\mathcal{E}(H) = \sum_{\sigma} |w_\sigma| \pi + \Delta(H) \pi. \quad (14)$$

Theorem 1 is proved in Section 2. There the quantity  $\sum_{\sigma} |w_\sigma| + \Delta(H)$  is related to an invariant of the (nonabelian) fundamental group  $\pi_1(S^2 - S, *)$ , where  $S$  is a set of representative points in  $S^2$  from four appropriately chosen sectors (the choice is determined by  $H$ ). Theorem 2 is proved in Section 3 by constructing a sequence of maps  $\nu_\epsilon \in H$  whose energy approaches the upper bound (13) as  $\epsilon$  approaches 0. The maps  $\nu_\epsilon$  are alternatively locally conformal or locally anticonformal except on a set whose area vanishes with  $\epsilon$ .

Theorems 1 and 2 follow from new methods compared to our previous work in [11, 12]. The derivation of the bounds (12) and (13) involve combinatorial-group-theoretic arguments and non-trivial explicit constructions. For conformal and anticonformal topologies, Corollary 1 coincides with results given in [11]. For nonconformal topologies, Corollary 1 is a sharp improvement of estimates obtained in [12], which are equivalent to  $\sum_{\sigma} |w_\sigma| \pi \leq \mathcal{E}(H) \leq 9 \sum_{\sigma} |w_\sigma| \pi$ .

## 1.1 Nematic liquid crystal configurations in a rectangular prism

In the Oseen-Frank theory [4, 21, 20], the local orientation of a nematic liquid crystal in a domain  $P \subset \mathbb{R}^3$  is described by a director field  $n : P \rightarrow RP^2$ . Equilibrium configurations are local minimizers of an energy functional with energy density quadratic in  $\nabla n$  and parameterised by three material-dependent constants. In the so-called one-constant approximation, the Oseen-Frank energy density reduces to the Dirichlet energy density  $(\nabla n)^2$ . For  $\Omega$  simply connected and  $n$  continuous in the interior of  $\Omega$ ,  $n$  may be assigned an orientation in a continuous way, and may be regarded as a continuous unit-vector field (ie,  $S^2$ -valued) on  $P$ . We assume this to be the case in what follows.

The equilibrium configurations depend crucially on the boundary conditions. For certain materials, *tangent boundary conditions* are appropriate, according to which  $n$  is required to be tangent to the boundary  $\partial P$ , which is assumed to be piecewise smooth. Let  $\mathcal{C}_T(P, S^2)$  denote the space of continuous unit-vector fields on  $P$  which satisfy tangent boundary conditions.

In a series of papers [10] – [16] we have studied the case where  $P$  is a polyhedral domain. One motivation are certain prototype designs for bistable liquid crystal displays, in which polygonal and polyhedral geometries support multiple equilibrium configurations with different optical properties [7]. The homotopy classification of  $\mathcal{C}_T(P, S^2)$  is described in [18, 14], and a lower bound for the infimum Dirichlet energy in terms of generalised minimal connections is obtained in [14]. For a review, see [16].

A number of results concern the case where  $P$  is a right rectangular prism,

$$P = \{r \in \mathbb{R}^3 \mid 0 \leq r_j \leq L_j\}. \quad (15)$$

For definiteness, we label the sides so that  $L_z \leq L_y \leq L_x$ . Let  $L = (L_x^2 + L_y^2 + L_z^2)^{1/2}$  denote the length of the prism diagonal. We have considered in particular *reflection-symmetric* homotopy classes in  $\mathcal{C}_T(P, S^2)$ . We say that a configuration  $n \in \mathcal{C}_T(P, S^2)$  is reflection-symmetric if it is invariant under reflection through the midplanes of  $P$ , ie

$$n(x, y, z) = n(L_x - x, y, z) = n(x, L_y - y, z) = n(x, y, L_z - z). \quad (16)$$

$n$  is therefore determined by its restriction to a fundamental domain with respect to reflections, eg

$$R = \{r \in \mathbb{R}^3 \mid 0 \leq r_j \leq \tfrac{1}{2}L_j\}. \quad (17)$$

A homotopy class  $h \subset \mathcal{C}_T(P, S^2)$  is reflection-symmetric if (and only if) it contains a reflection-symmetric representative. For reflection-symmetric homotopy classes, the infimum of the Dirichlet energy is given by

$$\mathcal{E}_3(h) = \inf_{n \in h} 8 \int_R (\nabla n)^2 dV. \quad (18)$$

Given  $n \in \mathcal{C}_T(P, S^2)$  and  $0 < a < L_z$ , we define a map  $\nu_{n,a} : O \rightarrow S^2$  by restricting  $n$  to the surface  $|r| = a$  in  $P$ , ie  $\nu_{n,a}(s) = n(as)$ . It is readily established that i)  $\nu_{n,a} \in \mathcal{C}_T(O, S^2)$ , ii) the homotopy class of  $\nu_{n,a}$  is independent of  $a$ , and iii) reflection-symmetric homotopy classes in  $\mathcal{C}_T(P, S^2)$  are in 1-1 correspondence with the homotopy classes of  $\mathcal{C}_T(O, S^2)$ . Let  $H$  denote the homotopy class of  $\mathcal{C}_T(O, S^2)$  corresponding to  $h$ .

Theorems 1 and 2 imply lower and upper bounds on  $\mathcal{E}_3(h)$  for reflection-symmetric homotopy classes  $h$ . Given  $\nu \in H \subset \mathcal{C}_T(O, S^2)$ , we construct a reflection-symmetric  $n \in \mathcal{C}_T(P, S^2)$  via  $n(r) = \nu(r/|r|)$  for  $r \in R$ . Then  $n \in h$ , and  $r^2|\nabla n|^2(r) = |\nu'|^2$ . The integral over  $R$  in (18) is bounded above by an integral over  $r < L/2$ , leading to the inequality  $\mathcal{E}_3(h) \leq 4L\mathcal{E}(H)$ . Conversely, for any  $n \in h$ , we have that  $r^2|\nabla n|^2 \geq |\nu'|^2$ . As the integral over  $R$  in (18) is bounded below by an integral over  $r < L_z/2$ , it follows that  $\mathcal{E}_3(h) \geq 4L_z\mathcal{E}(H)$ . We summarize these results in the following -

**Corollary 2.** *Let  $P$  be the right rectangular prism (15) with edge-lengths  $L_z \leq L_y \leq L_x$  and diagonal length  $L$ . Let  $h$  denote a reflection-symmetric homotopy class in  $\mathcal{C}_T(P, S^2)$ , and  $H$  the corresponding homotopy class in  $\mathcal{C}_T(O, S^2)$ . Then*

$$4L_z\mathcal{E}(H) \leq \mathcal{E}_3(h) \leq 4L\mathcal{E}(H). \quad (19)$$

For conformal and anticonformal homotopy classes, Corollary 2 coincides with the results of [11], and for nonconformal homotopy classes constitutes a sharp improvement of a result from [12].

Brezis, Coron and Lieb obtained the infimum Dirichlet energy for  $S^2$ -valued maps on  $\mathbb{R}^3$  with prescribed degrees on a set of excluded points, or defects (they also considered more general domains in  $\mathbb{R}^3$  with holes) [2]. Their result is expressed in terms of the length of a *minimal connection*, ie a pairing between defects of opposite sign. The estimates of [10] – [14] may be regarded as extensions of this classical result to the case of polyhedral domains with tangent boundary conditions, in which there are necessarily singularities at vertices. Our previous estimates may be expressed as a sum over minimal connections, one for each sector of the target  $S^2$  space, between the vertices of the polyhedral domain. The new lower bound in Theorem 1 contains additional topological information not captured by the minimal connection theory in [2]. In particular, the new homotopy invariant,  $\Delta(H)$  in (10), elucidates the fact that for certain nonconformal homotopy classes, the absolute number of pre-images of a regular value may necessarily be greater than the absolute value of  $|w_\sigma|$ . In such cases, the infimum energy is necessarily greater than the abelian bound,  $\pi \sum_\sigma |w_\sigma|$ , predicted by minimal connection theory. It would be interesting to generalise Corollary 2 to non-reflection-symmetric configurations on  $P$  as well as to more general polyhedral domains. Results in this direction may involve a nonabelian extension of the notion of minimal connection.

## 2 Lower bound for $\mathcal{E}(H)$

Given  $\nu \in \mathcal{C}_T(O, S^2) \cap W^{1,2}(O, S^2)$ , we can obtain a lower bound for the Dirichlet energy  $E(\nu)$  in terms of the number of preimages of a set of regular values of  $\nu$ , one from each sector of  $S^2$  (Lemma 2.1.1, Section 2.1). This leads to the following problem, which is addressed in Section 2.3: given a smooth unit-vector field  $\mu$  on the two-disk  $D^2$  for which the homotopy class of the boundary map  $\partial\mu$  is prescribed, find a lower bound for the number of preimages of a finite set  $S$  of regular values of  $\mu$ . The bound is expressed as the infimum of a certain function – the *spelling length* – over a product of conjugacy classes in the fundamental group  $\pi_1(S^2 - S, *)$ . The bound is obtained by analysing a simpler problem in Section 2.2, in which the target space is taken to be  $\mathbb{R}^2$  rather than  $S^2$ . The estimates of the spelling lengths relevant to our problem are given in Section 2.4, yielding a proof of Theorem 1.

Let us introduce some notation. Let  $X$  and  $Y$  be two-dimensional manifolds, possibly with boundary. Let  $\text{int}(X)$  denote the interior of  $X$  (similarly  $\text{int}(Y)$ ). Let  $f : X \rightarrow Y$  be piecewise continuously differentiable (in Section 3 it will be convenient to allow for maps with piecewise continuous derivatives). We say that  $y \in Y$  is a regular value of  $f$  if and only if  $y \in \text{int}(Y)$ ,  $f^{-1}(y) \subset \text{int}(X)$ , and  $f'$  is continuous and of full rank at each point of  $f^{-1}(y)$ . Let  $\mathcal{R}_f$  denote the set of regular values of  $f$ . We recall Sard's theorem [19], according to which  $\mathcal{R}_f$  has full Lebesgue measure. For  $y \in \mathcal{R}_f$ , let

$$d_f(y) = \sum_{x \in f^{-1}(y)} \text{sgn } \det f'(x), \quad (20)$$

$$D_f(y) = \sum_{x \in f^{-1}(y)} 1. \quad (21)$$

$d_f(y)$  is the algebraic degree, or simply the degree, of  $y$ , ie the number of preimages of  $y$  counted with orientation.  $D_f(y)$ , on the other hand, is the number of preimages of  $y$ . For convenience, we will refer to  $D_f(y)$  as the *absolute degree* of  $y$  although it should not be confused with the Hopf absolute degree [3] which is used elsewhere in the literature. We remark that  $d_f(y)$  is invariant under differentiable deformations of  $f$  (provided  $y$  remains a regular value), whereas  $D_f(y)$  is not. Clearly

$$|d_f(y)| \leq D_f(y). \quad (22)$$

Wrapping numbers are examples of algebraic degrees. Indeed, for  $\nu \in \mathcal{C}_T(O, S^2)$  differentiable and  $s_\sigma \in \Sigma_\sigma$  a regular value of  $\nu$ , we have that

$$d_\nu(s_\sigma) = -w_\sigma. \quad (23)$$

## 2.1 Lower bound and absolute degree

**Lemma 2.1.1.** *Let  $\nu \in \mathcal{C}_T(O, S^2)$  be differentiable. For each  $\sigma$ , let  $s_\sigma \in \Sigma_\sigma \cap \mathcal{R}_\nu$ . Then*

$$E(\nu) \geq \inf_{\{s_\sigma\}} \sum_{\sigma} D_\nu(s_\sigma) \pi. \quad (24)$$

*Proof.* From the inequality  $a^2 + b^2 + c^2 + d^2 \geq 2|ad - bc|$ , it follows that  $|\nu'|^2 \geq 2|\det \nu'|$ . Then

$$E(\nu) = \int_{p \in O} |\nu'|^2 dA_p \geq 2 \int_{p \in O} |\det \nu'| dA_p = 2 \int_{p \in O} |\det \nu'| \left( \sum_{\sigma} \int_{s \in \Sigma_\sigma} \delta_{S^2}(s, \nu(p)) dA_s \right) dA_p, \quad (25)$$

where  $\delta_{S^2}(s, t)$  is the Dirac delta function on  $S^2$  normalised to have unit integral. We may interchange the  $s$ - and  $p$ -integrals (this can be justified by introducing smoothed delta functions, appealing to Fubini's theorem, and taking the limit as the smoothing parameter goes to zero). For  $s \in \mathcal{R}_\nu$ , we have that

$$\int_{p \in O} |\det \nu'(p)| \delta_{S^2}(s, \nu(p)) dA_p = D_\nu(s). \quad (26)$$

By Sard's theorem, the set of regular values is of full measure. It follows from (25) and (26) that

$$E(\nu) \geq 2 \sum_{\sigma} \int_{s \in \mathcal{R}_{\nu} \cap \Sigma_{\sigma}} D_{\nu}(s) dA_s \geq \inf_{\{s_{\sigma}\}} \sum_{\sigma} D_{\nu}(s_{\sigma}) \pi, \quad (27)$$

as the sectors  $\Sigma_{\sigma}$  each have area  $\pi/2$ .  $\square$

## 2.2 Absolute degree of $\mathbb{R}^2$ -valued maps on $D^2$

Let  $D^2 \subset \mathbb{R}^2$  denote the unit disk with boundary  $\partial D^2 = S^1$ . Let  $R = \{y_1, \dots, y_n\}$  denote a set of  $n$  distinct points in  $\mathbb{R}^2$ . Let  $\pi_1(\mathbb{R}^2 - R, q)$  denote the fundamental group of the  $n$ -times punctured plane,  $\mathbb{R}^2 - R$ , based at  $q \in \mathbb{R}^2$ , where  $q \notin R$ .  $\pi_1(\mathbb{R}^2 - R, q)$  may be identified with the free group on  $n$  generators,  $F(c_1, \dots, c_n)$  (see, eg, [9]). We shall take the generator  $c_j$  to be the homotopy class of a loop  $\gamma_j$  based at  $q$  which encircles  $y_j$  once anticlockwise but encloses no other points of  $R$ . Equivalently,  $\gamma_j$  is freely homotopic in  $\mathbb{R}^2 - R$  to an  $\epsilon$ -circle about  $y_j$  oriented anticlockwise (with  $\epsilon$  small enough so that no other points of  $R$  are contained inside). It is straightforward to show that this condition determines  $c_j$  up to conjugacy. That is, if  $\gamma$  and  $\gamma'$  are two loops in  $\mathbb{R}^2 - R$  based at  $q$  which are freely homotopic to an anticlockwise-oriented  $\epsilon$ -circle about  $y_j$ , then

$$[\gamma'] = h[\gamma]h^{-1} \quad (28)$$

for some  $h \in \pi_1(\mathbb{R}^2 - R, q)$ .

Given  $g \in F(c_1, \dots, c_n)$  expressed as a product of the generators, the difference between the number of  $c_i$  and  $c_i^{-1}$  factors is well defined, and is called the *degree of  $c_i$  in  $g$* , and denoted by  $\deg_g(c_i)$ . Given  $g \in F(c_1, \dots, c_n)$ , we define a *spelling* to be a factorisation of  $g$  into a product of conjugated generators and inverse generators, eg

$$g = h_1 c_{i_1}^{\epsilon_1} h_1^{-1} \cdots h_r c_{i_r}^{\epsilon_r} h_r^{-1}, \quad (29)$$

where  $h_j \in F(c_1, \dots, c_n)$  and  $\epsilon_j = \pm 1$ . It is clear that

$$\sum_{s \mid i_s = j} \epsilon_s = \deg_g(c_j). \quad (30)$$

The number of factors in a spelling of  $g$ , (i.e.  $r$  in (29)), is not uniquely determined. We define the *spelling length* of  $g$ , denoted  $\Lambda(g)$ , to be the smallest possible number of factors amongst all spellings of  $g$ . From (30) it follows that the spelling length is determined modulo 2 by the sum of the degrees of the generators,

$$\Lambda(g) = \sum_{j=1}^n \deg_g(c_j) \pmod{2}, \quad (31)$$

and is bounded from below by the sum of their absolute values,

$$\Lambda(g) \geq \sum_{i=1}^n |\deg_g(c_i)|. \quad (32)$$



We refer to (32) as the abelian bound on the spelling length.

Let  $\phi : D^2 \rightarrow \mathbb{R}^2$  be differentiable, and let  $\partial\phi : S^1 \rightarrow \mathbb{R}^2 - \mathcal{R}_\phi$  denote the boundary map of  $\phi$ . Choose the points  $y_j$  above to be regular values of  $\phi$ , ie  $y_j \in \mathcal{R}_\phi$ , and take  $q$  to lie in the image of  $\partial\phi$ . We may regard  $\partial\phi$  as a loop in  $\mathbb{R}^2 - R$  based at  $q$ . We denote its homotopy class by  $[\partial\phi] \in \pi_1(\mathbb{R}^2 - R, q)$ . As the following shows, the spelling length of  $[\partial\phi]$  gives a lower bound on the cardinality of  $\phi^{-1}(R)$ .

**Proposition 2.2.1.** *Given  $\phi : D^2 \rightarrow \mathbb{R}^2$  smooth,  $R = \{y_1, \dots, y_n\} \subset \mathcal{R}_\phi$ , and  $\pi_1(\mathbb{R}^2 - R, q) \simeq F(c_1, \dots, c_n)$ , with generators  $c_j$  as above. Then*

$$\sum_{j=1}^n D_\phi(y_j) \geq \Lambda([\partial\phi]). \quad (33)$$

*Proof.* Let  $N = \sum_{j=1}^n D_\phi(y_j)$ , so that  $N$  is the number of points in  $\phi^{-1}(R)$ . Below we argue that  $[\partial\phi]$  can be expressed as a product of  $N$  factors,

$$[\partial\phi] = [\gamma_1] \cdots [\gamma_N], \quad (34)$$

in which each factor is conjugate to a generator or an inverse generator of  $F(c_1, \dots, c_n)$ . Then (34) constitutes a spelling of  $[\partial\phi]$  of length  $N$ , and (33) follows from the definition of the spelling length.

To establish the spelling (34), let  $\phi^{-1}(R) = \{x_1, \dots, x_N\}$ . We note that  $x_j$  is in the interior of  $D^2$ . Take  $p \in \partial D^2$  such that  $\phi(p) = q$ . We regard  $S^1 = \partial D^2$  as a loop based at  $p$ . As indicated in Figure 1, while keeping  $p$  fixed, we can continuously deform  $\partial D^2$  into a concatenation of  $N$  loops based at  $p$ , each of which encloses one of the  $x_a$ 's once (in the anticlockwise sense) and encloses none of the other  $x_a$ 's. The image of this deformation under  $\phi$  yields a homotopy from  $\partial\phi$  to a concatenation of  $N$  loops  $\gamma_a$  in  $\mathbb{R}^2 - R$  based at  $q$ , each of which is freely homotopic in  $\mathbb{R}^2 - R$  to an oriented  $\epsilon$ -circle about  $y_{j_a} = \phi(x_a)$ . From (28),  $[\gamma_a]$  is conjugate in  $\pi_1(\mathbb{R}^2 - R, q)$  to a generator  $c_j$  or an inverse generator  $c_j^{-1}$ , depending on the orientation of  $\gamma_a$ . □

For  $g = [\partial\phi]$ ,  $\deg_g(c_j)$  is equal to  $d_\phi(y_j)$ . Combining (32) and (33), we have the following sequence of inequalities

$$\sum_{j=1}^n D_\phi(y_j) \geq \Lambda([\partial\phi]) \geq \sum_{i=1}^n |d_\phi(y_j)|.$$

An example where the inequality is strict is  $g = c_1 c_2 c_1^{-1} c_2^{-1}$ ; in this case it is easy to show that  $\Lambda(g) = 2$  while  $\deg_g(c_1) = \deg_g(c_2) = 0$ .

### 2.3 Absolute degree of $S^2$ -valued maps on $D^2$

Let  $\mu : D^2 \rightarrow S^2$  be a differentiable  $S^2$ -valued map on  $D^2$  with boundary map  $\partial\mu : S^1 \rightarrow S^2$ . Let  $S = \{s_0, s_1, \dots, s_n\} \subset \mathcal{R}_\mu$  denote a set of  $n+1$  regular values of  $\mu$ . By analogy with Proposition 2.2.1, we seek a lower bound on the number of points in  $\mu^{-1}(S)$ . In contrast to Proposition 2.2.1, the bound we obtain will depend not only on the homotopy class of  $\partial\mu$ , but

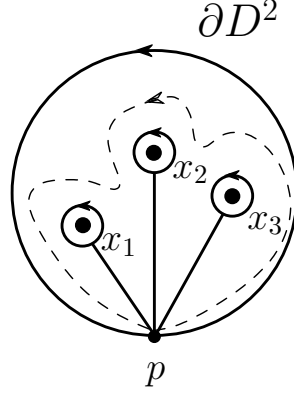


Figure 1: The boundary of the two-disk, regarded as a loop based at  $p$ , can be deformed into a concatenation of loops based at  $p$  encircling each of the preimages  $x_1, \dots, x_N$ .

also on the absolute and algebraic degrees of one of the  $s_j$ 's, which we fix to be  $s_0$ . The bound is obtained by excising a neighbourhood of  $\mu^{-1}(s_0)$  from  $D^2$  and defining an  $\mathbb{R}^2$ -valued map on the remainder to which Proposition 2.2.1 can be applied.

Let  $\Pi$  denote the projection from  $S^2 - \{s_0\}$  to  $\mathbb{R}^2$ , with  $s_0$  corresponding to the point at infinity. For  $1 \leq j \leq n$ , let  $r_j = \Pi(s_j)$ , and let  $R = \{r_1, \dots, r_n\}$ . Also, take  $u \in S^2$  in the image of  $\partial\mu$ , and let  $q = \Pi(u)$ . Then, since  $\Pi : S^2 - \{s_0\} \rightarrow \mathbb{R}^2$  is a diffeomorphism,

$$\pi_1(S^2 - S, u) \cong \pi_1(\mathbb{R}^2 - R, q) \cong F(c_1, \dots, c_n), \quad (35)$$

where, as in Section 2.2, the generator  $c_j$  is the homotopy class of an anticlockwise loop  $\gamma_j$  in  $\mathbb{R}^2 - R$  based at  $q$  which encloses  $r_j$  once and encloses none of the other  $r_k$ 's. Equivalently, we may regard  $c_j$  as the homotopy class of a loop  $\delta_j$  in  $S^2 - S$  based at  $u$  which separates  $s_j$  from the other  $s_k$ 's and is positively oriented with respect to  $s_j$ . Indeed, we may take  $\delta_j = \Pi^{-1}(\gamma_j)$ . In what follows, we regard  $[\partial\mu]$  as an element of  $F(c_1, \dots, c_n)$ .

Let  $\delta_0$  be a loop in  $S^2 - S$  based at  $u$  which separates  $s_0$  from the other  $s_j$ 's and is positively oriented with respect to  $s_0$ . Then  $\gamma_0 = \Pi(\delta_0)$  is a loop in  $\mathbb{R}^2$  based at  $q$  which encloses each of the  $r_j$ 's once in the clockwise sense. Let  $c_0 = [\gamma_0]$  denote its homotopy class. Then  $c_0$  may be expressed as a product of the  $c_j$ 's in which the sum of the exponents of each of the  $c_j$ 's is equal to  $-1$ .

We shall use the following notation. Given subsets  $V$  and  $W$  of a group  $G$ , we define their *set product*  $VW \subset G$  by

$$VW = \{vw \mid v \in V, w \in W\}. \quad (36)$$

We denote the  $n$ -fold product of  $V$  with itself by  $V^n$ . Clearly, if  $V$  and  $W$  are invariant under conjugation, ie  $hVh^{-1} = V$  for all  $h \in G$  and similarly for  $W$ , then  $VW$  is invariant under conjugation, in which case the set product is commutative, ie

$$VW = WV. \quad (37)$$

Given  $g \in G$ , let  $\langle g \rangle$  denote its conjugacy class, ie

$$\langle g \rangle = \{g' \in G \mid g' = hgh^{-1} \text{ for some } h \in G\}. \quad (38)$$

Clearly  $\langle g \rangle$  is invariant under conjugation, so the set product of conjugacy classes is commutative.

The following gives a lower bound for the number of points in  $\mu^{-1}(S)$ , given the absolute and algebraic degrees of  $s_0$ :

**Proposition 2.3.1.** *Let  $P = \frac{1}{2}(D_\mu(s_0) + d_\mu(s_0))$  and  $N = \frac{1}{2}(D_\mu(s_0) - d_\mu(s_0))$  denote the number of points in  $\mu^{-1}(s_0)$  with positive and negative orientation respectively. Let  $\langle c_0 \rangle$  denote the conjugacy class of  $c_0$  in  $F(c_1, \dots, c_n)$ , and let  $\mathcal{V}_{P,N} \subset F(c_1, \dots, c_n)$  be the set product given by*

$$\mathcal{V}_{P,N} = \{[\partial\mu]\} \langle c_0^{-1} \rangle^P \langle c_0 \rangle^N. \quad (39)$$

Then

$$\sum_{j=1}^n D_\mu(s_j) \geq \min_{g \in \mathcal{V}_{P,N}} \Lambda(g). \quad (40)$$

Thus, Proposition 2.3.1 implies that

$$\sum_{j=0}^n D_\mu(s_j) \geq D_\mu(s_0) + \min_{g \in \mathcal{V}_{P,N}} \Lambda(g). \quad (41)$$

*Proof.* As we show below, by excising a suitable neighbourhood of  $\mu^{-1}(s_0)$ , we can construct a differentiable map  $\mu_{P+N} : D^2 \rightarrow S^2$  such that

$$\begin{aligned} i) \quad & D_{\mu_{P+N}}(s_j) = D_\mu(s_j), \quad 1 \leq j \leq n, \\ ii) \quad & [\partial\mu_{P+N}] \in \mathcal{V}_{P,N}, \\ iii) \quad & \mu_{P+N}^{-1}(s_0) \text{ is empty.} \end{aligned} \quad (42)$$

In view of iii), the  $\mathbb{R}^2$ -valued map  $\phi = \Pi \circ \mu_{P+N}$  is differentiable on  $D^2$ , with i)  $D_\phi(r_j) = D_\mu(s_j)$  for  $1 \leq j \leq n$  and ii)  $[\partial\phi] \in \mathcal{V}_{P,N}$ . Then the claim (40) follows directly from Proposition 2.2.1, since

$$\sum_{j=1}^n D_\mu(s_j) = \sum_{j=1}^n D_\phi(r_j) \geq \Lambda([\partial\phi]) \geq \min_{g \in \mathcal{V}_{P,N}} \Lambda(g). \quad (43)$$

The construction of  $\mu_{P+N}$  proceeds inductively. For  $0 \leq i \leq P+N$ , we construct a differentiable map  $\mu_i : D^2 \rightarrow S^2$  such that

$$\begin{aligned} i) \quad & D_{\mu_i}(s_j) = D_\mu(s_j), \quad 1 \leq j \leq n, \\ ii) \quad & [\partial\mu_i] \in \mathcal{V}_{p_i, n_i}, \\ iii) \quad & \mu_i \text{ has } P - p_i \text{ (resp. } N - n_i) \text{ pre-images of } s_0 \text{ with positive (resp. negative) orientation,} \\ & \text{with } 0 \leq p_i \leq P, 0 \leq n_i \leq N \text{ and } p_i + n_i = i. \end{aligned} \quad (44)$$

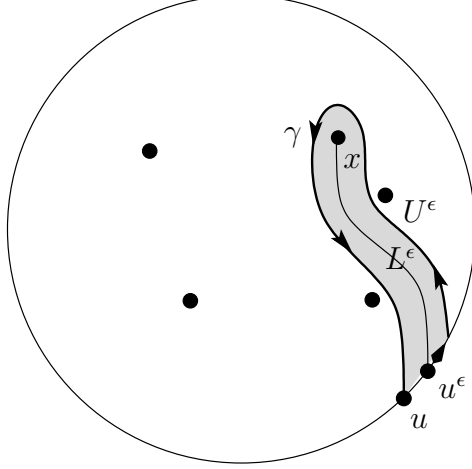


Figure 2:  $x$  is one of the points in  $\mu^{-1}(s_0)$ .  $U^\epsilon$  is the  $\epsilon$ -neighbourhood of a curve  $L^\epsilon$  from  $u^\epsilon$  to  $x$ , with  $\epsilon$  small enough so that  $x$  is the only point in  $\mu^{-1}(S)$  which lies in  $U^\epsilon$ .

For  $i = P + N$ , it is evident that  $\mu_{P+N}$  satisfies (42).

Here is the construction. For  $i = 0$ , we take  $\mu_0 = \mu$ , with  $p_0 = 0$  and  $n_0 = 0$ . Then  $\mu_0$  satisfies (44) trivially. Next, given  $\mu_i$  satisfying (44) with  $0 \leq i < P + N$ , we construct  $\mu_{i+1}$  as follows. Take  $x \in \mu_i^{-1}(s_0)$  and let  $\sigma = \text{sgn det } \mu'_i(x)$ . Take  $\epsilon > 0$  and take  $\tilde{u}^\epsilon$  to be the point on  $S^1 = \partial D^2$  at a distance  $\epsilon$  anticlockwise from  $\tilde{u}$ , where  $\partial\mu(\tilde{u}) = u$ . Let  $L^\epsilon$  be a non-self-intersecting differentiable curve from  $x$  to  $\tilde{u}^\epsilon$  which, apart from its endpoints, lies in the interior of  $D^2$  and contains no points in  $\mu_i^{-1}(S)$ . Choose  $\epsilon$  sufficiently small so that  $U^\epsilon$ , the open  $\epsilon$ -neighbourhood of  $L^\epsilon$ , contains no points in  $\mu_i^{-1}(S)$  other than  $x$ . See Figure 2. The boundary of  $U^\epsilon$ , oriented clockwise, may be regarded as a loop based at  $\tilde{u}$  which encloses a single point in  $\mu^{-1}(s_0)$  and encloses no other points in  $\mu^{-1}(S)$ . It follows that  $[\mu_i(\partial U^\epsilon)]$ , regarded as an element of  $F(c_1, \dots, c_n)$ , is conjugate to  $c_0^{-\sigma}$ .

The domain  $D^2 - U^\epsilon$  is homeomorphic to  $D^2$ ; let  $f : D^2 \rightarrow D^2 - U^\epsilon$  be a homeomorphism. We may take  $f$  to be a diffeomorphism on the interior of  $D^2$ . Take  $\mu_{i+1} = \mu_i \circ f$ . By construction and the induction hypothesis,  $D_{\mu_{i+1}}(s_j) = D_{\mu_i}(s_j) = D_\mu(s_j)$  for  $1 \leq j \leq n$ . Let  $P - p_{i+1}$  and  $N - n_{i+1}$  denote the number of points in  $\mu_{i+1}^{-1}(s_0)$  with positive and negative orientation respectively. By construction, if  $\sigma = 1$ , we have that  $p_{i+1} = p_i + 1$ ,  $n_{i+1} = n_i$ , while if  $\sigma = -1$ , we have that  $p_{i+1} = p_i$ ,  $n_{i+1} = n_i + 1$ . In either case, by induction,  $p_{i+1} + n_{i+1} = i + 1$ . Also by construction,  $\partial\mu_{i+1}$  is homotopic to the concatenation of  $\partial\mu_i$  and  $\mu_i(\partial U^\epsilon)$ . Since  $[\partial\mu_i] \in \mathcal{V}_{p_i, n_i}$  (by induction) and  $[\mu_i(\partial U^\epsilon)] \in \langle c_0 \rangle^{-\sigma}$ , it follows that  $[\partial\mu_{i+1}] \in \mathcal{V}_{p_i, n_i} \langle c_0 \rangle^{-\sigma} = \mathcal{V}_{p_{i+1}, n_{i+1}}$ . So  $\mu_{i+1}$  satisfies (44).  $\square$

There exist efficient algorithms for computing the spelling length [17]. However, we are not aware of general results for obtaining the minimum spelling length over a product of conjugacy classes. In the cases that arise in Section 2.4, we are nevertheless able to compute an effective lower bound for the spelling length on  $\mathcal{V}_{P, N}$  (cf Propositions 2.4.1 and 2.4.2).

## 2.4 Proof of Theorem 1

*Proof.* Let  $H$  be a homotopy class in  $\mathcal{C}_T(O, S^2)$ , with invariants  $(e, k, \Omega)$  and  $\{w_\sigma\}$ . Given  $\nu \in H$ , we show that

$$E(\nu) \geq \sum_{\sigma} |w_{\sigma}| \pi + \Delta(H) \pi. \quad (45)$$

Using arguments from [11], one can show that differentiable maps are dense in  $\mathcal{C}_T(O, S^2) \cap W^{1,2}(O, S^2)$ . Therefore, we may assume that  $\nu$  is differentiable. For each  $\sigma$ , choose  $s_{\sigma} \in \mathcal{R}_{\nu} \cap \Sigma_{\sigma}$ . Then from Lemma 2.1.1, it suffices to show that for all  $s_{\sigma} \in \mathcal{R}_{\nu} \cap \Sigma_{\sigma}$ , we have the inequality

$$\sum_{\sigma} (D_{\nu}(s_{\sigma}) - |w_{\sigma}|) \geq \Delta(H). \quad (46)$$

Since  $D_{\nu}(s_{\sigma}) \geq |d_{\nu}(s_{\sigma})|$  (cf (22)) and  $d_{\nu}(s_{\sigma}) = -w_{\sigma}$  (cf (23)), (46) follows immediately for  $H$  conformal or anticonformal (cf (8)). For  $H$  nonconformal, (46) is equivalent to (cf (10))

$$\sum_{\sigma} (D_{\nu}(s_{\sigma}) - |w_{\sigma}|) \geq 2w_{\sigma_+} - 2 \sum_{\sigma \sim \sigma_+} \Phi(w_{\sigma}) - 2\chi, \quad (47a)$$

$$\sum_{\sigma} (D_{\nu}(s_{\sigma}) - |w_{\sigma}|) \geq 2|w_{\sigma_-}| - 2 \sum_{\sigma \sim \sigma_-} \Phi(-w_{\sigma}) - 2\chi. \quad (47b)$$

Without loss of generality, we may assume that the edge signs are all equal to +1, ie

$$e_x = e_y = e_z = +1. \quad (48)$$

(This follows from noting that the Dirichlet energy is invariant under reflection in, for example, the  $xy$ -plane of the target space. That is, if  $\nu = (\nu_x, \nu_y, \nu_z)$  and  $\nu' = (\nu_x, \nu_y, -\nu_z)$ , then  $E(\nu') = E(\nu)$ . Under reflection in the  $xy$ -plane, the edge signs transform as  $(e_x, e_y, e_z) \mapsto (e_x, e_y, -e_z)$ . Similarly,  $e_x$  and  $e_y$  change sign under reflections in the  $yz$ - and  $zx$ -coordinate planes respectively.) With (48), the expression (5) for the wrapping numbers becomes

$$w_{\sigma} = \frac{1}{4\pi} \Omega + \frac{1}{8} + \frac{1}{2} \sum_j \sigma_j k_j - \delta_{\sigma, (+++)} \quad (49)$$

We proceed to prove (47). The argument divides into several cases according to the signs of the  $k_j$ 's. We shall consider one representative case in detail, namely where all the  $k_j$ 's are positive. The arguments for the remaining cases are then briefly sketched. For definiteness, and without loss of generality, we assume that  $k_x \leq k_y \leq k_z$ .

*Case 1.*  $k_x, k_y, k_z > 0$ . In view of (48), we have that  $\chi = 0$ . We consider the bound (47a) first. Since  $D_{\nu}(s_{\sigma}) \geq |w_{\sigma}|$ , (47a) is obviously implied by

$$D_{\nu}(s_{\sigma_+}) - |w_{\sigma_+}| + \sum_{\sigma \sim \sigma_+} (D_{\nu}(s_{\sigma}) - |w_{\sigma}|) \geq 2w_{\sigma_+} - 2 \sum_{\sigma \sim \sigma_+} \Phi(w_{\sigma}), \quad (50)$$

in which the sector sum on the left-hand side is restricted to  $\sigma_+$  and the sectors adjacent to  $\sigma_+$ . (It turns out that these are the only sectors in which  $D_\nu(s_\sigma)$  is, in certain cases, necessarily greater than  $|w_\sigma| = |d_\nu(s_\sigma)|$ .) From (49), we may take  $\sigma_+ = (+++)$ . The sectors adjacent to  $\sigma_+$  are then  $(-++)$ ,  $(+ - +)$  and  $(++-)$ . To simplify the notation, we replace  $(+++)$ ,  $(-++)$ ,  $(+ - +)$  and  $(++-)$  by 0, 1, 2 and 3 respectively. We let  $S = \{s_0, s_1, s_2, s_3\}$ , and denote a generic point in  $S$  by  $s_j$ .

If we identify  $O$  with the unit disk  $D^2$ , we may identify  $\nu$  with an  $S^2$ -valued map  $\mu$  on  $D^2$ . We proceed to apply Proposition 2.3.1 to obtain a lower bound on  $\sum_j D_\nu(s_j)$ . For this we need to calculate generators for the fundamental group of  $S^2 - S$  based at a point  $u$  in the image of the boundary of  $\mu$ , and to express  $[\partial\mu]$  in terms of them. For definiteness, we take  $u = \hat{\mathbf{x}}$  ( $\hat{\mathbf{x}}$  belongs to the image of  $\mu$  since, by assumption,  $e_x = 1$ ).

The loops we consider are sequences of quarter-arcs of great circles between the coordinate unit vectors  $E = \{\pm\hat{\mathbf{x}}, \pm\hat{\mathbf{y}}, \pm\hat{\mathbf{z}}\}$ . We will denote these loops as follows. Given  $e, e' \in E$  with  $e \neq -e'$ , let  $(e, e')$  denote the quarter-arc of the great circle from  $e$  to  $e'$  if  $e \neq e'$ , and the null arc at  $e$  if  $e = e'$ . Given  $e_i \in E$  with  $e_i \neq -e_{i+1}$ , let  $(e_1, \dots, e_n)$  denote the curve composed of the sequence  $(e_1, e_2), (e_2, e_3), \dots, (e_{n-1}, e_n)$ . Curves can be concatenated in the obvious way, ie  $(e_1, \dots, e_m, \dots, e_n) = (e_1, \dots, e_m)(e_m, \dots, e_n)$ . We let  $(e_1, \dots, e_n)^i$  denote the curve  $(e_1, \dots, e_n)$  concatenated with itself  $i$  times. In this notation, the boundary of  $\mu$ , regarded as a loop in  $S^2$  based at  $\hat{\mathbf{x}}$ , is given by

$$\partial\mu = C_z^{k_z}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) C_x^{k_x}(\hat{\mathbf{y}}, \hat{\mathbf{z}}) C_y^{k_y}(\hat{\mathbf{z}}, \hat{\mathbf{x}}), \quad (51)$$

where

$$C_x = (\hat{\mathbf{y}}, -\hat{\mathbf{z}}, -\hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{y}}), \quad C_y = (\hat{\mathbf{z}}, -\hat{\mathbf{x}}, -\hat{\mathbf{z}}, \hat{\mathbf{x}}, \hat{\mathbf{z}}), \quad C_z = (\hat{\mathbf{x}}, -\hat{\mathbf{y}}, -\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{x}}) \quad (52)$$

describe the great circles about the  $x$ -,  $y$ - and  $z$ -axes.

Let

$$\delta_0 = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{x}}), \quad \delta_1 = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, -\hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{y}}, \hat{\mathbf{x}}), \quad \delta_2 = (\hat{\mathbf{x}}, \hat{\mathbf{z}}, -\hat{\mathbf{y}}, \hat{\mathbf{x}}), \quad \delta_3 = (\hat{\mathbf{x}}, -\hat{\mathbf{z}}, \hat{\mathbf{y}}, \hat{\mathbf{x}}). \quad (53)$$

It is easily verified that  $\delta_j$  is a loop based at  $\hat{\mathbf{x}}$  which traverses the boundary of  $\Sigma_j$  once with positive orientation, and therefore separates  $s_j$  from the other  $s_k$ 's and encloses  $s_j$  with positive orientation. Let  $c_j \in \pi_1(S^2 - S, \hat{\mathbf{x}})$  denote the homotopy class of  $\delta_j$ .

As discussed in Section 2.3,  $\pi_1(S^2 - S, \hat{\mathbf{x}}) \cong F(c_1, c_2, c_3)$ . Straightforward calculation yields the following expressions for  $c_0$  and  $[\partial\mu]$  in terms of the generators  $c_1$ ,  $c_2$  and  $c_3$ :

$$c_0 = c_3^{-1} c_1^{-1} c_2^{-1} = (c_2 c_1 c_3)^{-1}, \quad [\partial\mu] = c_3^{k_z-1} c_1^{k_x-1} c_2^{k_y-1}. \quad (54)$$

Note that, since  $k_j > 0$  by assumption, the exponents  $k_j - 1$  are nonnegative.

Applying Proposition 2.3.1 to  $\mu$ , we get that

$$\sum_{j=1}^3 D_\mu(s_j) = \sum_{j=1}^3 D_\nu(s_j) \geq \min_{g \in \mathcal{V}_{P,N}} \Lambda(g), \quad (55)$$

where

$$P = \frac{1}{2}(D_\nu(s_0) + d_\nu(s_0)), \quad N = \frac{1}{2}(D_\nu(s_0) - d_\nu(s_0)) \quad (56)$$

and the minimum is taken over

$$g \in \{c_3^{k_z-1} c_1^{k_x-1} c_2^{k_y-1}\} \langle c_2 c_1 c_3 \rangle^P \langle (c_2 c_1 c_3)^{-1} \rangle^N. \quad (57)$$

The following combinatorial-group-theoretic result implies a bound on  $\Lambda(g)$  in (55):

**Proposition 2.4.1.** *Let  $\mathcal{P}_{p,n} \subset F(A, B, C)$  be the set product given by*

$$\mathcal{P}_{p,n} = \langle A^i B^j C^k \rangle \langle CBA \rangle^p \langle (CBA)^{-1} \rangle^n. \quad (58)$$

Then for  $g \in \mathcal{P}_{p,n}$ ,

$$\Lambda(g) \geq i + j + k - (p + n). \quad (59)$$

Thus, for example, the minimum spelling length of words of the form  $f_1(ABC)f_1^{-1}f_2(CBA)^{-1}f_2^{-1}$ , where  $f_i \in F(A, B, C)$ , is equal to 2 (we can apply Proposition 2.4.1 to this example by taking  $i = j = k = 1$  and  $p = 0, n = 1$ ). This can be seen directly as follows: A spelling of length 2 is obtained by taking  $f_1 = e$  and  $f_2 = A$  to get  $(ABC)A(A^{-1}B^{-1}C^{-1})A^{-1} = h_1 C h_1^{-1} h_2 C^{-1} h_2^{-1}$ , where  $h_1 = AB$  and  $h_2 = A$ . By (31), the minimum spelling length is either 2 or 0, and a spelling of length zero cannot be found as  $ABC$  and  $CBA$  belong to different conjugacy classes in  $F(A, B, C)$ . Note that (32) gives the abelian lower bound

$$\min_{g \in \mathcal{P}_{p,n}} \Lambda(g) \geq i + j + k + 3(p - n), \quad (60)$$

which for  $n \leq 2p$  already implies Proposition 2.4.1. Thus, Proposition 2.4.1 is stronger than the abelian bound (60) for

$$n > 2p \quad (61)$$

and therefore, requires independent proof in this case. The proof of Proposition 2.4.1 is given in the Appendix. In fact, we believe the following result holds,

$$\min_{g \in \mathcal{P}_{p,n}} \Lambda(g) \geq i + j + k - n, \quad (62)$$

which is stronger than Proposition 2.4.1 for  $p > 0$ . However, for the purposes of Theorem 1, Proposition 2.4.1 is sufficient.

From (55), (57) and Proposition 2.4.1, it follows that

$$D_\nu(s_0) + \sum_{j=1}^3 D_\nu(s_j) \geq k_x + k_y + k_z - 3, \quad (63)$$

where we have used  $P + N = D_\nu(s_0)$ . Using (49), we can express the right-hand side above in terms of the wrapping numbers,

$$k_x + k_y + k_z - 3 = 3w_0 - (w_1 + w_2 + w_3). \quad (64)$$

Substituting (64) into (63) (and recalling that  $w_0 = w_{(++++)} > 0$ ), we get that

$$D_\nu(s_0) - |w_0| + \sum_{j=1}^3 (D_\nu(s_j) - |w_j|) \geq 2w_0 - \sum_{j=1}^3 (w_j + |w_j|) = 2w_0 - 2 \sum_{j=1}^3 \Phi(w_j), \quad (65)$$

which is just the required bound (47a).

The bound (47b) is obtained from a similar argument. From (49),  $\sigma_- = (- - -)$ , so that the sectors adjacent to  $\sigma_-$  are  $(+ - -)$ ,  $(- + -)$ , and  $(- - +)$ . To simplify the notation, we replace  $(- - -)$ ,  $(+ - -)$ ,  $(- + -)$ , and  $(- - +)$  by 0, 1, 2 and 3 respectively, and let  $S = \{s_0, s_1, s_2, s_3\}$ . As above, we argue that (47b) is implied by

$$D_\nu(s_0) - |w_0| + \sum_{j=1}^3 (D_\nu(s_j) - |w_j|) \geq 2w_0 - 2 \sum_{j=1}^3 \Phi(-w_j). \quad (66)$$

We proceed to employ Proposition 2.3.1 to obtain a lower bound on  $\sum_{j=1}^3 D_\nu(s_j)$ . We introduce loops  $\delta_j$  based at  $\hat{\mathbf{x}}$ ,

$$\delta_0 = (x \ y - x - z - y - x \ y \ x), \quad \delta_1 = (x - y - z \ x), \quad \delta_2 = (x - z - x \ y - z \ x), \quad \delta_3 = (x \ z - x - y \ z \ x). \quad (67)$$

One can verify that  $\delta_j$  separates  $s_j$  from the other  $s_k$ 's and encloses  $s_j$  with positive orientation. Let  $c_j \in \pi_1(S^2 - S, \hat{\mathbf{x}})$  denote the homotopy classes of  $\delta_j$ . As above, we identify  $\nu$  with an  $S^2$ -valued map  $\mu$  on  $D^2$ . Calculation gives

$$c_0 = c_2^{-1} c_1^{-1} c_3^{-1}, \quad [\partial\mu] = c_3^{-k_z} c_1^{-k_x} c_2^{-k_y}. \quad (68)$$

Then Proposition 2.3.1 together with Proposition 2.4.1 yield (47b).

Case 2.  $k_x, k_y > 0, k_z < 0$ . From (49),  $\sigma_+$  and  $\sigma_-$  are given by  $(+ + -)$  and  $(- - +)$ , respectively, and the calculation proceeds as in Case 1 with one key difference. In the expressions analogous to (55), the spelling length  $\Lambda(g)$  is to be minimised over words  $g$  of a slightly different form to that of Proposition 2.4.1 (ie, it is the conjugacy classes of  $ABC$  and its inverse which appear, rather than those of  $CBA$ ). By analogy with Proposition 2.4.1, we have the following:

**Proposition 2.4.2.** *Let  $\mathcal{Q}_{p,n} \subset F(A, B, C)$  be the set product given by*

$$\mathcal{Q}_{p,n} = \langle A^i B^j C^k \rangle \langle ABC \rangle^p \langle (ABC)^{-1} \rangle^n. \quad (69)$$

*Then for  $g \in \mathcal{Q}_{p,n}$ ,*

$$\Lambda(g) \geq i + j + k - (p + n + 2). \quad (70)$$

Thus, for example, the minimum spelling length of words of the form  $ABChC^{-1}B^{-1}A^{-1}h^{-1}$  for  $h \in F(A, B, C)$ , is obviously zero. The contribution  $-2$  in (70) accounts for the term  $\chi = 1$  in  $\Delta(H)$  in this case (cf (9) and (10)).



Case 3.  $k_x > 0, k_y, k_z < 0$ . From (49),  $\sigma_+$  and  $\sigma_-$  are given by  $(+ - -)$  and  $(- + +)$  respectively, and the calculation proceeds as in Case 1.

Case 4.  $k_x, k_y, k_z < 0$ . From (49),  $\sigma_+$  and  $\sigma_-$  are given by  $(- - -)$  and  $(+ + +)$  respectively, and the calculation proceeds as in Case 2.

Case 5.  $k_x k_y k_z = 0$ . If one of the  $k_j$ 's vanishes, then in most cases,  $\sigma_+$  and one of its adjacent sectors will share the same largest wrapping number, and similarly for  $\sigma_-$ . Then  $\Delta(H) = 0$ , and (46) follows automatically. Exceptions can arise when  $k_x = k_y$  and  $k_z = 0$ , but it is straightforward to verify (47) in this special case [13]; the argument is omitted.  $\square$

### 3 Upper bound for $\mathcal{E}(H)$

Given a homotopy class  $H \subset \mathcal{C}_T(O, S^2)$ , we can construct an explicit representative  $\nu \in H$  that realizes the upper bound in Theorem 2. As stated in Section 1, homotopy classes in  $\mathcal{C}_T(O, S^2)$  are classified as being either conformal, anticonformal or nonconformal. Conformal and anticonformal homotopy classes are studied in detail in [11, 12]. For these homotopy classes,  $\Delta(H) = 0$  by definition (see (10)) and consequently, the infimum Dirichlet energy is bounded from below by  $\mathcal{E}(H) \geq \sum_{\sigma} |w_{\sigma}|$  i.e. the lower bound is simply the abelian bound in (32). In [11], we construct explicit conformal (anticonformal) representatives  $\nu$  for conformal (anticonformal) homotopy classes such that for a regular value  $\mathbf{s}_{\sigma} \in \Sigma_{\sigma} \cap \mathcal{R}_{\nu}$ , the absolute degree coincides with the absolute value of the wrapping number i.e.  $D_{\nu}(\mathbf{s}_{\sigma}) = |w_{\sigma}|$  for all  $\sigma$  and the corresponding Dirichlet energy is

$$E(\nu) = \pi \sum_{\sigma} D_{\nu}(\mathbf{s}_{\sigma}) = \pi \sum_{\sigma} |w_{\sigma}|. \quad (71)$$

Thus, these representatives achieve the upper bound in Theorem 2.

Nonconformal homotopy classes have also been studied in some detail in [12]. In [12], we construct explicit representatives  $\nu$  in nonconformal homotopy classes from a juxtaposition of conformal and anticonformal configurations. The representative  $\nu$  is taken to be either conformal or anticonformal almost everywhere in  $O$  except for a small interior disc. We insert a certain number,  $N \geq 1$ , of full coverings of  $S^2$  with either positive or negative orientation within this interior disc. The choice of  $N$  and the orientation of these full coverings (positive or negative) clearly depends on the nonconformal homotopy class in question. One can show that the representative  $\nu$ , thus defined, has Dirichlet energy strictly greater than the upper bound in Theorem 2.

In this section, we return to the upper bound problem for nonconformal homotopy classes. We construct alternative explicit representatives by introducing quarter-sphere configurations. We take the representative  $\nu$  to be either conformal or anticonformal everywhere away from the vertices of  $O$ . Near the vertices of  $O$ , we modify  $\nu$  and insert quarter-sphere configurations. The quarter-sphere configurations are either conformal or anticonformal configurations that cover a pair of adjacent octants with either negative or positive orientation and preserve the tangent boundary conditions. The quarter-sphere configurations allow us to realize the minimal number of pre-images

consistent with (12) and hence, saturate the lower bound in Theorem 1 and realize the upper bound in Theorem 2. In Section 3.1, we consider a simple illustrative example. In Section 3.2, we review the main results for conformal and anticonformal topologies from [11] and formally define quarter-sphere configurations. In Section 3.3, we explicitly define the representatives for nonconformal homotopy classes and in Section 3.4, we carry out the relevant energy estimates that yield a proof for Theorem 2.

### 3.1 An Example

Let  $H$  be the nonconformal homotopy class defined by the invariants

$$\begin{aligned} e_j &= +1, \quad k_j = +1, \quad \forall j \\ \Omega &= \frac{3\pi}{2}. \end{aligned} \tag{72}$$

The corresponding wrapping numbers are shown below (5) -

$$\begin{aligned} w_{+++} &= w_{-++} = w_{+-+} = w_{++-} = 1 \\ w_{+--} &= w_{-+-} = w_{--+} = 0 \\ w_{---} &= -1. \end{aligned} \tag{73}$$

From Theorem 1,  $\mathcal{E}(H) \geq 7\pi$  and  $\Delta(H) = 2$  so that there necessarily exists an octant  $\Sigma_\sigma$ , with  $\sigma \sim (- - -)$ , such that a regular value  $\mathbf{s}_\sigma \in \Sigma_\sigma$  has at least two pre-images in spite of the fact that  $w_\sigma = 0$  (or equivalently  $d_\nu(\mathbf{s}_\sigma) = 0$ ).

We construct a representative  $\nu \in H$  on the following lines. Let  $O_z$  denote a small neighbourhood of the vertex  $\hat{\mathbf{z}}$  in  $O$ . We take  $\nu$  to be an anticonformal configuration on  $O \setminus O_z$  with wrapping numbers as shown below

$$\begin{aligned} w_{+++} &= w_{-++} = w_{+-+} = w_{++-} = w_{--+} = 1 \\ w_\sigma &= 0 \quad \text{otherwise.} \end{aligned} \tag{74}$$

In  $O_z$ , we modify  $\nu$  to insert a quarter-sphere configuration. This quarter-sphere configuration is a conformal configuration by construction and its image covers the pair of adjacent octants,  $\Sigma_{--+}$  and  $\Sigma_{---}$ , exactly once with negative orientation. The representative,  $\nu$ , thus defined from the juxtaposition of the anticonformal configuration in (74) and the quarter-sphere configuration, has the correct topology  $H$  in (72), satisfies the tangent boundary conditions and is continuous everywhere in  $O$ . The corresponding absolute degrees for regular values  $\mathbf{s}_\sigma \in \mathcal{R}_\nu \cap \Sigma_\sigma$  are

$$\begin{aligned} D_\nu(\mathbf{s}_\sigma) &= 1 \quad \text{for } \sigma = \{(+++), (-++), (+-+), (++-)\} \\ D_\nu(\mathbf{s}_\sigma) &= 2 \quad \text{for } \sigma = (- - +) \\ D_\nu(\mathbf{s}_\sigma) &= 1 \quad \text{for } \sigma = (- - -). \end{aligned} \tag{75}$$

Since  $\nu$  is either conformal or anticonformal almost everywhere by construction, we can explicitly estimate its Dirichlet energy from (71)

$$\mathcal{E}(H) \leq E(\nu) = \pi \sum_{\sigma} D_\nu(\mathbf{s}_\sigma) = 7\pi, \tag{76}$$

consistent with the upper bound in Theorem 2.

For the sake of comparison, we briefly outline the construction of a representative  $\nu \in H$ , following the methods in [12]. Let  $D_\epsilon$  denote a small interior disc of radius  $0 < \epsilon < \frac{1}{8}$ . In [12], we take  $\nu$  to be an anticonformal configuration on  $O \setminus D_\epsilon$  with wrapping numbers -

$$\begin{aligned} w_{+++} &= w_{-++} = w_{+-+} = w_{+--} = 2 \\ w_{+--} &= w_{-+-} = w_{--+} = 1 \\ w_{---} &= 0 \end{aligned} \tag{77}$$

and in  $D_\epsilon$ , we insert a conformal configuration that covers  $S^2$  exactly once with negative orientation. The absolute degrees in this case are given by

$$\begin{aligned} D_\nu(\mathbf{s}_\sigma) &= 3 \quad \sigma = \{(+++), (-++), (+-+), (++-)\} \\ D_\nu(\mathbf{s}_\sigma) &= 2 \quad \sigma = \{(+--), (-+-), (- -+)\} \\ D_\nu(\mathbf{s}_\sigma) &= 1 \quad \sigma = (---) \quad \text{where } \mathbf{s}_\sigma \in \Sigma_\sigma \cap \mathcal{R}_\nu. \end{aligned} \tag{78}$$

The corresponding Dirichlet energy is

$$\mathcal{E}(H) \leq E(\nu) = \pi \sum_{\sigma} D_\nu(\mathbf{s}_\sigma) = 19\pi, \tag{79}$$

which is more than twice the upper bound in Theorem 2.

This example demonstrates how a quarter-sphere configuration can enable us to realize the upper bound in Theorem 2 and realize the minimum number of pre-images consistent with Theorem 1. More generally, we need a sequence of alternating conformal and anticonformal quarter-sphere configurations localized near the vertices of  $O$  and these quarter-sphere configurations are chosen carefully so as to preserve the topology and the tangent boundary conditions. Explicit details are given in the subsequent sections.

### 3.2 Complex representation

For a given homotopy class  $H \subset \mathcal{C}_T(O, S^2)$ , we represent the representative  $\nu \in H$  by a complex-valued function using stereographic projection. Let  $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$  denote the extended complex plane. Let  $P : S^2 \rightarrow \mathbb{C}^*$  denote the stereographic projection of the unit sphere into the extended complex plane with  $-\hat{\mathbf{z}}$  being projected to  $\infty$  i.e. for  $\mathbf{e} = (e_x, e_y, e_z) \in S^2$ ,  $P(\mathbf{e})$  is given by

$$P(\mathbf{e}) = \frac{e_x + ie_y}{1 + e_z}. \tag{80}$$

We let  $Q = P(O) \subset \mathbb{C}^*$  denote the projected domain; then  $Q$  is the quarter-disc given by

$$Q = \left\{ w = \rho e^{i\phi} \mid 0 \leq \rho \leq 1, 0 \leq \phi \leq \frac{\pi}{2} \right\}. \tag{81}$$

The boundary of  $Q$  consists of three segments,  $\partial Q = C_1 \cup C_2 \cup C_3$  - (i) the real segment  $C_1 = \{w \in \mathbb{R}; 0 \leq w \leq 1\}$ , which is the projection of the  $zx$ -edge of  $O$ , (ii) the imaginary segment  $C_2 =$

$\{w = it; 0 \leq t \leq 1\}$ , which is the projection of the  $yz$ -edge of  $O$  and (iii) the quarter-circle  $C_3 = \{w = e^{i\phi}; 0 \leq \phi \leq \frac{\pi}{2}\}$ , which is the projection of the  $xy$ -edge of  $O$ . The vertices of  $Q$  are at the points  $P(\hat{\mathbf{x}}) = 1$ ,  $P(\hat{\mathbf{y}}) = i$  and  $P(\hat{\mathbf{z}}) = 0$  respectively.

Given  $\nu : O \rightarrow S^2$ , we define the corresponding projected map  $K : Q \rightarrow \mathbb{C}^*$  by

$$K = P \circ \nu \circ P^{-1}. \quad (82)$$

Then if  $\nu = (\nu_x, \nu_y, \nu_z)$ , we have that

$$K(w) = \frac{\nu_x(\mathbf{s}) + i\nu_y(\mathbf{s})}{1 + \nu_z(\mathbf{s})} \quad \text{where } \mathbf{s} = P^{-1}(w). \quad (83)$$

Let  $\mathcal{C}_T(Q, \mathbb{C}^*)$  denote the space of maps  $K : Q \rightarrow \mathbb{C}^*$  for which  $\nu \in \mathcal{C}_T(O, S^2)$ . The tangent boundary conditions require that (i)  $K(w)$  is real if  $w$  is real (i.e.  $w \in C_1$ ) (ii)  $K(w)$  is imaginary if  $w$  is imaginary (for  $w \in C_2$ ) and (iii)  $|K(w)| = 1$  if  $|w| = 1$  (for  $w \in C_3$ ). Finally, if  $\nu$  is differentiable, then so is  $K$  and the Dirichlet energy of  $\nu$  is given in terms of  $K$  as shown below -

$$E(\nu) = E(K) = \int_Q \mathcal{H}(K) d^2w \quad (84)$$

where

$$\mathcal{H}(K) = 4 \left( \frac{|\partial_w K|^2 + |\partial_{\bar{w}} K|^2}{(1 + |K|^2)^2} \right) \quad (85)$$

is the Dirichlet energy density in complex coordinates.

### 3.2.1 Conformal and anticonformal representatives

We briefly review the main results in [11, 12] for conformal and anticonformal homotopy classes. In [12], we show that a homotopy class  $H \subset \mathcal{C}_T(O, S^2)$  is conformal (anticonformal) if and only if it admits a conformal (anticonformal) representative. Under the stereographic projection  $P : S^2 \rightarrow \mathbb{C}^*$  defined in (80), we represent a conformal representative in a conformal homotopy class by an analytic function  $f : Q \rightarrow \mathbb{C}^*$ . The tangent boundary conditions means that if  $w$  is a zero of  $f$ , then so are  $\pm \bar{w}$  and  $-w$  and  $\frac{1}{w}$  is a pole. These constraints along with the conditions that  $f(0) = 0$  or  $f(0) = \infty$  (since  $\nu(\hat{\mathbf{z}}) = \pm \hat{\mathbf{z}}$ ) and  $f(1) = \pm 1$  (since  $\nu(\hat{\mathbf{x}}) = \pm \hat{\mathbf{x}}$ ) imply that  $f(w)$  is a rational function of the form [11]

$$f(w) = \pm w^{2m+1} \prod_{j=1}^a \left( \frac{w^2 - r_j^2}{r_j^2 w^2 - 1} \right)^{\rho_j} \prod_{k=1}^b \left( \frac{w^2 + s_k^2}{s_k^2 w^2 + 1} \right)^{\sigma_k} \times \prod_{l=1}^c \left( \frac{(w^2 - t_l^2)(w^2 - \bar{t}_l^2)}{(t_l^2 w^2 - 1)(\bar{t}_l^2 w^2 - 1)} \right)^{\tau_l}. \quad (86)$$

The  $r_j$ 's denote the real zeros ( $\rho_j = 1$ ) and poles ( $\rho_j = -1$ ) of  $f$  between 0 and 1; the  $s_k$ 's, the imaginary zeros and poles of  $f$  (according to whether  $\sigma_k = \pm 1$ ) between 0 and  $i$ ; and the  $t_l$ 's, the complex

zeros and poles of  $f$  (according to whether  $\tau_l = \pm 1$ ) with modulus less than one and argument between 0 and  $\pi/2$ ,  $m$  is any integer and  $a, b$  and  $c$  are non-negative integers. The homotopy invariants  $(e, k, \Omega)$  can be explicitly computed in terms of the parameters  $\{m, a, b, c, r_j, s_k, \tau_l, \rho_j, \sigma_k, \tau_l\}$ . Analogous remarks apply to anticonformal representatives, with complex analytic functions being replaced by complex antianalytic functions  $f(\bar{w})$ , where  $f$  is rational and of the form (86).

For  $H = (e, k, \Omega)$  conformal or anticonformal, let  $F_H$  denote the corresponding conformal/anticonformal representative of the form (86). For such representatives  $F_H$ ,  $D_{F_H}(\mathbf{s}_\sigma)$  is independent of the choice of the regular value  $\mathbf{s}_\sigma \in \Sigma_\sigma$  and  $D_{F_H}(\mathbf{s}_\sigma) = |w_\sigma|$  for all  $\sigma$ . The corresponding Dirichlet energy can be explicitly computed as in (71) and

$$E(F_H) = \pi \sum_{\sigma} D_{F_H}(\sigma) = \pi \sum_{\sigma} |w_\sigma|. \quad (87)$$

Therefore,  $E(F_H)$  realizes the lower bound of Theorem 1 and the upper bound in Theorem 2 i.e.

$$E(F_H) = \mathcal{E}(H) \quad (88)$$

as required.

### 3.2.2 Quarter-sphere configurations

Quarter-sphere configurations are defined on small neighbourhoods of the vertices of  $Q$ . For concreteness, let

$$Q_\epsilon = \{w \in Q; \rho = |w| \leq \epsilon\} \subset Q$$

denote the closed  $\epsilon$ -neighbourhood of the vertex  $w = 0$  where  $\epsilon > 0$ . Let

$$\rho_0 = 0 \text{ and } \rho_m = \epsilon^{L+1-m} \text{ with } 1 \leq m \leq L, \quad (89)$$

where  $L$  is a positive integer. We note that

$$\frac{\rho_{m+1}}{\rho_m} = \frac{1}{\epsilon}.$$

We partition  $Q_\epsilon$  into  $(2L - 1)$  concentric quarter-annuli - (i) the quarter-sphere configurations are defined on the annuli

$$2\rho_{m-1} \leq \rho \leq \rho_m \quad 1 \leq m \leq L \quad (90)$$

and (ii) we interpolate between the different quarter-sphere configurations on the intervening annuli

$$\rho_n \leq \rho \leq 2\rho_n \quad 1 \leq n \leq L - 1. \quad (91)$$

Consider the quarter-annuli for  $1 \leq m \leq L$ . We define the quarter-sphere configurations  $g_{m,\epsilon}$  as shown below -

$$g_{m,\epsilon}(w) = \begin{cases} -\frac{w}{\sqrt{\epsilon\rho_m}}, & 2\rho_{m-1} \leq \rho \leq \rho_m, m \text{ odd}, \\ \frac{\rho_{m-1}}{\sqrt{\epsilon w}}, & 2\rho_{m-1} \leq \rho \leq \rho_m, m \text{ even}. \end{cases} \quad (92)$$

We consider the case of  $m$  odd first. For  $m$  odd,  $g_{m,\epsilon}$  is a rational analytic function (conformal configuration) which is real on the real axis and imaginary on the imaginary axis i.e.  $g_{m,\epsilon}$  satisfies the tangent boundary conditions on the real and imaginary axes as required. One can directly verify that the image of  $g_{m,\epsilon}$  covers the pair of adjacent octants  $\Sigma_{--\pm}$  exactly once with negative orientation, except for a small neighbourhood of  $\pm\hat{\mathbf{z}}$  on  $S^2$ <sup>1</sup>. For regular values  $\xi_\sigma$  not contained in these excluded neighbourhoods i.e. for  $\xi_\sigma \in \mathcal{R}_{g_{m,\epsilon}}$  satisfying

$$|\xi_\sigma|, \frac{1}{|\xi_\sigma|} < \frac{1}{\sqrt{\epsilon}}, \quad (93)$$

we have that

$$d_{g_{m,\epsilon}}(\xi_\sigma) = \begin{cases} -1, & \sigma = (- - \pm), \\ 0, & \text{otherwise} \end{cases} \quad (94)$$

where  $d_{g_{m,\epsilon}}(\xi_\sigma)$  is the algebraic degree defined in (20). The corresponding Dirichlet energy is easily estimated using (84) and (85) and we have the following -

$$E(g_{m,\epsilon}) = \int_{2\rho_{m-1} \leq \rho \leq \rho_m} \mathcal{H}(g_{m,\epsilon}) \, d^2w = 2\pi \left( \frac{1 - 4\epsilon^2}{(1 + \epsilon)(1 + 4\epsilon)} \right) \quad (95)$$

and

$$E(g_{m,\epsilon}) \leq 2\pi + C\epsilon \quad (96)$$

where  $C$  is a positive constant independent of  $\epsilon$ .

The case of  $m$  even can be treated in an analogous manner. Here,  $g_{m,\epsilon}$  is a rational antianalytic function (anticonformal configuration) that is real on the real axis and imaginary on the imaginary axis. Again, one can directly verify that the image of  $g_{m,\epsilon}$  covers the pair of adjacent octants  $\Sigma_{++\pm}$  exactly once with positive orientation except for a small neighbourhood of  $\pm\hat{\mathbf{z}}$  on  $S^2$  and for regular values  $\xi_\sigma$  satisfying (93), the algebraic degrees are given by

$$d_{g_{m,\epsilon}}(\xi_\sigma) = \begin{cases} +1, & \sigma = (+ + \pm), \\ 0, & \text{otherwise.} \end{cases} \quad (97)$$

The corresponding Dirichlet energy is estimated as in (95) and we have that

$$E(g_{m,\epsilon}) = \int_{2\rho_{m-1} \leq \rho \leq \rho_m} \mathcal{H}(g_{m,\epsilon}) \, d^2w \leq 2\pi + C\epsilon \quad (98)$$

for a positive constant  $C$  independent of  $\epsilon$ .

On the annuli  $\rho_n \leq \rho \leq 2\rho_n$  with  $1 \leq n \leq L - 1$ , we define the interpolatory functions  $h_{n,\epsilon}$  according to

$$h_{n,\epsilon}(w) = \begin{cases} ((1 - s_n(\rho))/g_{n,\epsilon}(w) + s_n(\rho)/g_{n+1,\epsilon}(w))^{-1}, & n \text{ odd}, \\ (1 - s_n(\rho))g_{n,\epsilon} + s_n(\rho)g_{n+1,\epsilon}(w), & n \text{ even} \end{cases} \quad (99)$$

---

<sup>1</sup>Strictly speaking, the image of  $g_{m,\epsilon}$  covers the projected octants  $P(\Sigma_{--\pm})$  on  $\mathbb{C}^*$  except for a  $\sqrt{\epsilon}$ -neighbourhood of  $w = 0$  and  $\infty$  but here and in what follows, we do not explicitly distinguish between the octants and their projection on the complex plane.

where  $s_n$  is the switching function

$$s_n(\rho) = \frac{\rho - \rho_n}{\rho_n}, \quad \rho_n \leq \rho \leq 2\rho_n. \quad (100)$$

It is easy to verify that  $h_{n,\epsilon}$ , thus defined, is real on the real axis and imaginary on the imaginary axis since the  $g_{n,\epsilon}$ 's satisfy tangent boundary conditions on the real and imaginary axes. The energy estimates for  $h_{n,\epsilon}$  can be easily carried out. We consider the case of  $n$  odd in (99) first with  $g_{n,\epsilon} = -\frac{w}{\sqrt{\epsilon\rho_n}}$ ,  $g_{n+1,\epsilon} = \frac{\rho_n}{\sqrt{\epsilon w}}$ . For  $s_n$  in (100), we have that

$$|\partial_w s_n|^2 + |\partial_{\bar{w}} s_n|^2 < \frac{1}{\rho_n^2}. \quad (101)$$

Similarly, we note that

$$\left| \partial_w \frac{1}{h_{n,\epsilon}} \right|^2 \leq C \left( \left| \partial_w \left( \frac{1}{g_{n,\epsilon}} \right) \right|^2 + |\partial_w s_n|^2 \left( \left| \frac{1}{g_{n,\epsilon}} \right|^2 + \left| \frac{1}{g_{n+1,\epsilon}} \right|^2 \right) \right) \leq C' \frac{\epsilon}{\rho_n^2} \quad (102)$$

and likewise

$$\left| \partial_{\bar{w}} \frac{1}{h_{n,\epsilon}} \right|^2 \leq C \left( \left| \partial_{\bar{w}} \left( \frac{1}{g_{n+1,\epsilon}} \right) \right|^2 + |\partial_{\bar{w}} s_n|^2 \left( \left| \frac{1}{g_{n,\epsilon}} \right|^2 + \left| \frac{1}{g_{n+1,\epsilon}} \right|^2 \right) \right) \leq C'' \frac{\epsilon}{\rho_n^2} \quad (103)$$

where  $C$ ,  $C'$  and  $C''$  are positive constants independent of  $\epsilon$ . On the other hand

$$\left( 1 + \left| \frac{1}{h_{n,\epsilon}} \right|^2 \right) \geq 1 \quad (104)$$

so that

$$\mathcal{H} \left( \frac{1}{h_{n,\epsilon}} \right) \leq D \frac{\epsilon}{\rho_n^2} \quad (105)$$

from (85), for a positive constant  $D$  independent of  $\epsilon$ . Substituting the above into (84), we obtain the following -

$$E \left( \frac{1}{h_{n,\epsilon}} \right) \leq C_1 \epsilon, \quad n \text{ odd} \quad (106)$$

for a positive constant  $C_1$  independent of  $\epsilon$  and since  $\mathcal{H} \left( \frac{1}{h_{n,\epsilon}} \right) = \mathcal{H}(h_{n,\epsilon})$ , we have that

$$E(h_{n,\epsilon}) \leq C_1 \epsilon. \quad (107)$$

We repeat the same calculations for the case  $n$  even and it can be shown that the energy estimate (107) holds for all  $n$  i.e.

$$E(h_{n,\epsilon}) \leq C_2 \epsilon, \quad 1 \leq n \leq L-1 \quad (108)$$

where  $C_2$  is a positive constant independent of  $\epsilon$ .

From Lemma 2.1.1, we have that the Dirichlet energy of  $h_{n,\epsilon}$  is bounded from below by

$$E(h_{n,\epsilon}) \geq 2 \sum_{\sigma} \int_{\xi_{\sigma} \in \mathcal{R}_{h_{n,\epsilon}} \cap P(\Sigma_{\sigma})} D_{h_{n,\epsilon}}(\xi_{\sigma}) d^2 w. \quad (109)$$

Since  $E(h_{n,\epsilon}) \rightarrow 0$  as  $\epsilon \rightarrow 0$  and  $D_{h_{n,\epsilon}}(\xi_{\sigma}) \geq |d_{h_{n,\epsilon}}(\xi_{\sigma})|$ , we deduce that

$$d_{h_{n,\epsilon}}(\xi_{\sigma}) = 0 \quad 1 \leq n \leq L-1. \quad (110)$$

for all regular values  $\xi_{\sigma}$  satisfying (93), apart from a subset whose measure vanishes with  $\epsilon$  and for all  $\sigma$ .

We define the configuration  $\Gamma_{L,\epsilon} : Q_{\epsilon} \rightarrow \mathbb{C}^*$  as follows -

$$\Gamma_{L,\epsilon}(w) = \begin{cases} g_{m,\epsilon}(w), & 2\rho_{m-1} \leq \rho \leq \rho_m, \\ h_{n,\epsilon}(w), & \rho_n \leq \rho \leq 2\rho_n \end{cases} \quad (111)$$

where  $1 \leq m \leq L$  and  $1 \leq n \leq L-1$ .

**Proposition 3.2.1.** *The function  $\Gamma_{L,\epsilon} : Q_{\epsilon} \rightarrow \mathbb{C}^*$  defined in (111) has the following properties -*

- (i)  $\Gamma_{L,\epsilon}$  is real (imaginary) on the real (imaginary) axis i.e.  $\Gamma_{L,\epsilon}$  satisfies the tangent boundary conditions on the real and imaginary axes,
- (ii) the Dirichlet energy is bounded from above by

$$E(\Gamma_{L,\epsilon}) \leq 2\pi L + C_3 \epsilon \quad (112)$$

where  $C_3$  is a positive constant independent of  $\epsilon$ ,

- (iii) the algebraic degrees are given by

$$d_{\Gamma_{L,\epsilon}}(\xi_{\sigma}) = W_{\sigma}(L) \quad (113)$$

for regular values  $\xi_{\sigma}$  satisfying (93) and  $W_{\sigma}(L)$  is defined as shown below

$$W_{\sigma}(L) = \begin{cases} -\left[\frac{L+1}{2}\right], & \sigma = (- - \pm), \\ \left[\frac{L}{2}\right], & \sigma = (+ + \pm), \\ 0, & \text{otherwise.} \end{cases} \quad (114)$$

*Proof.* Property (i) is immediate from the definition of  $\Gamma_{L,\epsilon}$  in (111), since  $g_{m,\epsilon}$  and  $h_{n,\epsilon}$  satisfy the tangent boundary conditions on the real and imaginary axes. To show (112), we note that

$$\int_{Q_{\epsilon}} \mathcal{H}(\Gamma_{L,\epsilon}) d^2 w = \sum_{m=1}^L \int_{2\rho_{m-1} \leq \rho \leq \rho_m} \mathcal{H}(g_{m,\epsilon}) d^2 w + \sum_{n=1}^{L-1} \int_{\rho_n \leq \rho \leq 2\rho_n} \mathcal{H}(h_{n,\epsilon}) d^2 w, \quad (115)$$

and substitute the energy estimates (96), (98) and (108) into (115), yielding the upper bound (112).

To show (113), we observe that for  $L$  even, we have an equal number of conformal and anticonformal quarter-sphere configurations whereas for  $L$  odd, we have  $(L+1)/2$  conformal quarter-sphere



configurations and  $(L - 1)/2$  anticonformal quarter-sphere configurations (refer to (92)). The algebraic degree,  $d_{\Gamma_{L,\epsilon}}(\xi_\sigma)$ , of a regular value  $\xi_\sigma$  is given by

$$d_{\Gamma_{L,\epsilon}}(\xi_\sigma) = \sum_{m=1}^L d_{g_{m,\epsilon}}(\xi_\sigma) + \sum_{n=1}^{L-1} d_{h_{n,\epsilon}}(\xi_\sigma). \quad (116)$$

It suffices to substitute (94), (97) and (110) into (116) and (113) immediately follows.  $\square$

*Note:* We note that  $d_{\Gamma_{L,\epsilon}}(\xi_\sigma)$  is independent of the choice of regular value  $\xi_\sigma$  and only depends on  $\sigma$ ; therefore, these algebraic degrees are referred to as  $d_{\Gamma_{L,\epsilon}}(\sigma)$  in the subsequent sections.

### 3.2.3 Symmetries and translations of quarter-spheres

Let  $\gamma : \mathbb{C}^* \rightarrow \mathbb{C}^*$  denote the Möbius transformation

$$\gamma(w) = \frac{i - w}{i + w}. \quad (117)$$

It is easy to verify that  $\gamma(1) = i$ ,  $\gamma(i) = 0$ ,  $\gamma(0) = 1$  and  $\gamma(Q) = Q$ . We define

$$\gamma_x = \gamma, \quad \gamma_y = \gamma^2, \quad \gamma_z = \gamma^3 (= \text{id}). \quad (118)$$

Then

$$Q_{j,\epsilon} = \gamma_j(Q_\epsilon) \quad (119)$$

is the closed  $\epsilon$ -neighbourhood of  $\gamma_j(0)$  in  $Q$ ; for example,  $Q_{x,\epsilon}$  is a closed  $\epsilon$ -neighbourhood of the vertex  $w = 1$ ,  $Q_{y,\epsilon}$  is a closed  $\epsilon$ -neighbourhood of the vertex  $w = i$  and  $Q_{z,\epsilon} = Q_\epsilon$ .

Let  $M_j \geq 1$  be a positive integer. We define  $G_{j,M_j,\epsilon} : Q_{j,\epsilon} \rightarrow \mathbb{C}^*$  by

$$G_{j,M_j,\epsilon} = \gamma_j \circ \Gamma_{M_j,\epsilon} \circ \gamma_j^{-1} \quad (120)$$

For  $\sigma = (\sigma_x \sigma_y \sigma_z)$ , we define the following permutations

$$p_x(\sigma) = (\sigma_z \sigma_x \sigma_y), \quad p_y(\sigma) = (\sigma_y \sigma_z \sigma_x) \text{ and } p_z(\sigma) = (\sigma_x \sigma_y \sigma_z). \quad (121)$$

Then for  $G_{x,M_x,\epsilon}$ , the octants  $\Sigma_{--\pm}$  and  $\Sigma_{++\pm}$  are mapped onto  $\Sigma_{p_x(--\pm)} = \Sigma_{\pm--}$  and  $\Sigma_{p_x(++\pm)} = \Sigma_{\pm++}$  respectively. Therefore, the conformal quarter-sphere configurations in  $G_{x,M_x,\epsilon}$  ( $m$  odd in (92)) cover the octant pair  $\Sigma_{\pm--}$  exactly once with negative orientation and the anticonformal quarter-sphere configurations ( $m$  even in (92)) cover the octant pair  $\Sigma_{\pm++}$  exactly once with positive orientation. Similarly, for  $G_{y,M_y,\epsilon}$ , the octants  $\Sigma_{--\pm}$  and  $\Sigma_{++\pm}$  are mapped onto  $\Sigma_{p_y(--\pm)} = \Sigma_{-\pm-}$  and  $\Sigma_{p_y(++\pm)} = \Sigma_{+\pm+}$  respectively. Therefore, the conformal quarter-sphere configurations cover the octant pair  $\Sigma_{-\pm-}$  exactly once with negative orientation and the anticonformal quarter-sphere configurations cover the octant pair  $\Sigma_{+\pm+}$  exactly once with positive orientation.

It then follows directly from Proposition 3.2.1 that

$$G_{j,M_j,\epsilon}(w) \text{ satisfies the tangent boundary conditions on } Q_{j,\epsilon} \cap \partial Q, \quad (122)$$

$$\int_{Q_{j,\epsilon}} \mathcal{H}(G_{j,M_j,\epsilon}) d^2w \leq 2M_j\pi + C\epsilon, \quad (123)$$

$$d_{G_{j,M_j,\epsilon}}(\xi_\sigma^j) = W_{p_j(\sigma)}(M_j), \quad (124)$$

where  $C$  is a positive constant independent of  $\epsilon$ ,  $\xi_\sigma^j$  is a regular value,  $\xi_\sigma = \gamma_j^{-1}(\xi_\sigma^j)$  satisfies (93) and  $W_{p_j(\sigma)}(M_j)$  has been defined in (114).

### 3.3 Explicit representatives

In this section, we construct explicit representatives  $\nu$  for all nonconformal homotopy classes. The general construction procedure is as follows. Let  $0 < \epsilon < \frac{1}{8}$ . We partition the domain  $Q$  into four subdomains - (i)  $Q_{x,2\epsilon}$ , (ii)  $Q_{y,2\epsilon}$ , (iii)  $Q_{z,2\epsilon}$  and (iv)  $Q_0 = Q \setminus \cup_j Q_{j,2\epsilon}$  which we refer to as the *bulk* domain. Given a nonconformal homotopy class  $H$ , we specify a set of three non-negative integers  $M = (M_x, M_y, M_z)$  and a conformal or anticonformal homotopy class  $H_0 = (e_0, k_0, \Omega_0)$  with edge signs  $e_0 = (e_{0x}, e_{0y}, e_{0z})$  given by

$$e_{0j} = (-1)^{M_j}. \quad (125)$$

Given  $H_0$ , there exists a complex rational representative  $F_{H_0}$  of the form (86) with this topology (see [11, 12]). On each of the subdomains  $Q_{j,2\epsilon}$ , we insert  $M_j$  quarter-sphere configurations. The quarter-sphere configurations are explicitly given by (92) and (120) and we interpolate between the different quarter-sphere configurations with negligible energy as in (108). Given  $F_{H_0}$  and the different quarter-sphere configurations on the sub-domains  $Q_{j,2\epsilon}$ , we define an overall configuration  $K_{H_0,M,\epsilon} : Q \rightarrow \mathbb{C}^*$  as shown below -

$$K_{H_0,M,\epsilon}(w) = \begin{cases} F_{H_0}(w), & w \in Q_0, \\ G_{j,M_j,\epsilon}(w), & w \in Q_{j,\epsilon}, M_j > 0, \\ ((1-s)/G_{j,M_j,\epsilon} + s/F_{H_0})^{-1}(w), & w \in Q_{j,2\epsilon} - Q_{j,\epsilon}, M_j > 0, M_j \text{ odd}, \\ ((1-s)G_{j,M_j,\epsilon} + sF_{H_0})(w), & w \in Q_{j,2\epsilon} - Q_{j,\epsilon}, M_j > 0, M_j \text{ even}. \end{cases} \quad (126)$$

Here  $s$  is the switching function on  $Q_{2\epsilon} - Q_\epsilon$  given by

$$s(w) = \frac{\rho - \epsilon}{\epsilon} \quad (127)$$

and  $G_{j,M_j,\epsilon}$  has been defined in (120). If  $M_j = 0$  for some  $j$ , then  $K_{H_0,M,\epsilon} = F_{H_0}$  on  $Q_0 \cup Q_{j,2\epsilon}$ . We point out that the functions  $((1-s)/G_{j,M_j,\epsilon} + s/F_{H_0})^{-1}$  and  $((1-s)G_{j,M_j,\epsilon} + sF_{H_0})$  interpolate between  $G_{j,M_j,\epsilon}(w)$  and  $F_{H_0}$  on the annular strip  $Q_{j,2\epsilon} - Q_{j,\epsilon}$ . The representative  $\nu$  is then taken to be the inverse projection of  $K_{H_0,M,\epsilon}$  i.e.  $\nu = P^{-1}(K_{H_0,M,\epsilon})$ .

Let  $H = (e, k, \Omega)$  denote an arbitrary nonconformal homotopy class. As discussed in Section 2, we can, without loss of generality, take the edge signs to be

$$e_j = +1 \quad \forall j. \quad (128)$$

For concreteness, we also assume that the  $k_j$ 's are ordered as follows -  $0 < |k_x| \leq |k_y| \leq |k_z|$ . As in Section 2, we focus on one representative case  $k_j > 0$  for all  $j$ ; the details for the remaining cases are sketched briefly. With  $e_j = +1$  and  $k_j > 0$  for all  $j$ , the corresponding wrapping numbers are given by (5) i.e.

$$w_\sigma = \frac{1}{4\pi}\Omega + \frac{1}{2}\sum_j \sigma_j k_j + \left(\frac{1}{8} - \delta_{\sigma, +++}\right). \quad (129)$$

It is easily verified from (129) that the  $w_\sigma$ 's are ordered as follows -

$$w_{+++} \geq w_{-++} \geq w_{+-+} \geq w_{++-}, w_{--+} \geq w_{-+-} \geq w_{+--} \geq w_{---}. \quad (130)$$

$H$  is nonconformal for  $1 \leq w_{+++} \leq k_x + k_y + k_z - 2$ . In this case,  $w_{+++} > 0$  is the largest positive wrapping number and  $w_{---} < 0$  is the smallest negative wrapping number. We consider two different cases according to whether  $w_{--+} - w_{++-} = k_z - (k_x + k_y) < 0$  or  $w_{--+} - w_{++-} = k_z - (k_x + k_y) \geq 0$ . For convenience, we let  $n = w_{+++}$ , where  $n \in [1, k_x + k_y + k_z - 2]$  for  $H$  nonconformal. Each case above is further divided into sub-cases according to the value of  $n$  and for each sub-case, we explicitly specify the representative  $\nu$  in terms of  $M = (M_x, M_y, M_z)$  and a conformal or anticonformal topology  $H_0$ .

**Case 1:**  $k_z - (k_x + k_y) < 0$

**Case 1a:**  $1 \leq n \leq k_y - 1$ :

The bulk topology  $H_0$ :

$$\begin{aligned} e_0 &= (e_{0x}, e_{0y}, e_{0z}) \text{ where } e_{0x} = e_{0y} = e_{0z} = 1 \\ k_0 &= (k_{0x}, k_{0y}, k_{0z}) \text{ where } k_{0x} = k_x, \quad k_{0y} = k_y - n, \quad k_{0z} = k_z - n \\ \Omega_0 &= -2\pi \sum_j k_j + \frac{7\pi}{2} + 4n\pi. \end{aligned} \quad (131)$$

The number of quarter-sphere configurations are given by

$$M_x = 2n, \quad M_y = M_z = 0. \quad (132)$$

**Case 1b:**  $k_y \leq n \leq \frac{\sum_j k_j - 2}{2}$ :

The bulk topology  $H_0$ :

$$\begin{aligned} e_{0x} &= e_{0y} = e_{0z} = 1 \\ k_{0x} &= 1, \quad k_{0y} = 1, \quad k_{0z} = \sum_j k_j - 2n - 2 \\ \Omega_0 &= -2\pi \sum_j k_j + \frac{7\pi}{2} + 4n\pi. \end{aligned} \quad (133)$$

The number of quarter-sphere configurations are given by

$$M_x = 2(n - k_x + 1), \quad M_y = 2(n - k_y + 1), \quad M_z = 2(k_x + k_y - n - 2). \quad (134)$$

**Case 1c:**  $\frac{\sum_j k_j - 1}{2} \leq n \leq k_x + k_y - 2$ :

The bulk topology  $H_0$ :

$$\begin{aligned} e_{0x} &= e_{0y} = e_{0z} = 1 \\ k_{0x} &= 0, \quad k_{0y} = 0, \quad k_{0z} = 2n + 2 - \sum_j k_j \end{aligned} \quad (135)$$

$$\Omega_0 = -2\pi \sum_j k_j + \frac{7\pi}{2} + 4n\pi \quad (136)$$

The number of quarter-sphere configurations are given by

$$M_x = 2(k_y + k_z - n - 1), \quad M_y = 2(k_x + k_z - n - 1), \quad M_z = 2(n - k_z + 1). \quad (137)$$

**Case 1d:**  $k_x + k_y - 1 \leq n \leq k_x + k_z - 2$ :

The bulk topology  $H_0$ :

$$\begin{aligned} e_{0x} &= e_{0y} = -1, \quad e_{0z} = 1 \\ k_{0x} &= 0, \quad k_{0y} = 0, \quad k_{0z} = 2n + 3 - \sum_j k_j \end{aligned} \quad (138)$$

$$\Omega_0 = -2\pi \sum_j k_j + \frac{11\pi}{2} + 4n\pi \quad (139)$$

The number of quarter-sphere configurations are given by

$$M_x = 2(k_y + k_z - n - 2) + 1, \quad M_y = 2(k_x + k_z - n - 2) + 1, \quad M_z = 2(n - k_z + 1). \quad (140)$$

**Case 1e:**  $k_x + k_z - 1 \leq n \leq k_y + k_z - 2$ :

The bulk topology  $H_0$ :

$$\begin{aligned} e_{0x} &= e_{0y} = -1, \quad e_{0z} = 1 \\ k_{0x} &= 0, \quad k_{0y} = n - k_z + 1, \quad k_{0z} = n - k_x - k_y + 2 \end{aligned} \quad (141)$$

$$\Omega_0 = -2\pi \sum_j k_j + \frac{11\pi}{2} + 4n\pi. \quad (142)$$

The number of quarter-sphere configurations are given by

$$M_x = 2(k_y + k_z - n - 2) + 1, \quad M_y = 2k_x - 1, \quad M_z = 0. \quad (143)$$

**Case 1f:**  $k_y + k_z - 1 \leq n \leq \sum_j k_j - 2$ :

The bulk topology  $H_0$ :

$$\begin{aligned} e_{0x} &= -1, \quad e_{0y} = e_{0z} = 1 \\ k_{0x} &= k_x, \quad k_{0y} = n - k_x - k_z + 1, \quad k_{0z} = n - k_x - k_y + 1 \end{aligned} \quad (144)$$

$$\Omega_0 = -2\pi \sum_j k_j + \frac{9\pi}{2} + 4n\pi. \quad (145)$$

The number of quarter-sphere configurations are given by

$$M_x = 2 \left( \sum_j k_j - n - 2 \right) + 1, \quad M_y = M_z = 0. \quad (146)$$

**Case 2:**  $k_z - (k_x + k_y) \geq 0$

**Case 2a:**  $1 \leq n \leq k_y - 1$ : Take  $H_0$  and  $M = (M_x, M_y, M_z)$  as in Case 1a.

**Case 2b:**  $k_y \leq n \leq k_x + k_y - 2$ : Take  $H_0$  and  $M = (M_x, M_y, M_z)$  as in Case 1b.

**Case 2c:**  $k_x + k_y - 1 \leq n \leq k_z - 1$ : This case is slightly different to the remaining cases discussed in this section. Firstly, we note that there are precisely four non-negative wrapping numbers i.e.  $w_\sigma \geq 0$  for  $\sigma = (\pm \pm +)$  and four non-positive wrapping numbers i.e.  $w_\sigma \leq 0$  for  $\sigma = (\pm \pm -)$  and  $\Delta(H) = 0$  for nonconformal topologies in this range (see (10)).

As in the preceding cases, we specify the representative  $\nu$  in terms of a bulk topology  $H_0$  and three non-negative integers  $M = (M_x, M_y, M_z)$ . The bulk topology  $H_0$  is anticonformal with invariants

$$\begin{aligned} e_{0x} &= 1, \quad e_{0y} = -1, \quad e_{0z} = 1 \\ k_{0x} &= k_{0y} = k_{0z} = 0 \\ \Omega_0 &= \frac{\pi}{2}. \end{aligned} \quad (147)$$

We take

$$\begin{aligned} M_x &= 2k_y + 2(k_z - n - 1) \\ M_y &= 2(n - k_y) + 1 \\ M_z &= 0. \end{aligned} \quad (148)$$

For the sub-domains  $Q_{x,\epsilon}$  and  $Q_{y,\epsilon}$ , we define modified configurations  $G'_{x,M_x,\epsilon}$  and  $G''_{y,M_y,\epsilon}$  as follows. We consider  $G'_{x,M_x,\epsilon}$  first. Let  $1 \leq m \leq M_x$ . On the quarter annuli,  $2\rho_{m-1} \leq \rho \leq \rho_m$ , we define

$$g'_{m,\epsilon}(w) = \begin{cases} -\frac{w}{\sqrt{\epsilon}\rho_m}, & 1 \leq m \leq 2(k_z - n - 1), m \text{ odd}, \\ \frac{\rho_{m-1}}{\sqrt{\epsilon}w}, & 1 \leq m \leq 2(k_z - n - 1), m \text{ even}, \\ g_{m,\epsilon}(w), & 2(k_z - n - 1) < m \leq M_x \end{cases} \quad (149)$$

where  $\rho_m$  has been defined in (89) and  $g_{m,\epsilon}$  in (92). For  $1 \leq m \leq 2(k_z - n - 1)$  and  $m$  odd,  $g'_{m,\epsilon}$  covers the pair of adjacent octants,  $\Sigma_{--\pm}$ , exactly once with negative orientation whereas for  $m$  even,  $g'_{m,\epsilon}$  covers the pair of adjacent octants,  $\Sigma_{+-\pm}$ , exactly once with negative orientation. For  $m > 2(k_z - n - 1)$ ,  $g'_{m,\epsilon}$  coincides with  $g_{m,\epsilon}$ .  $\Gamma'_{M_x,\epsilon}$  and  $G'_{x,M_x,\epsilon}$  are defined in terms of  $g'_{m,\epsilon}$  by analogy with (111) and (120) respectively.

By analogy with Proposition 3.2.1, we can show that  $G'_{x,M_x,\epsilon}$  satisfies the tangent boundary conditions on  $Q_{x,\epsilon} \cap \partial Q$  and the corresponding Dirichlet energy is bounded from above by

$$E\left(G'_{x,M_x,\epsilon}\right) = \int_{Q_{x,\epsilon}} \mathcal{H}\left(G'_{x,M_x,\epsilon}\right) d^2w \leq 2\pi M_x + C\epsilon \quad (150)$$

where  $C$  is a positive constant independent of  $\epsilon$ . Let

$$W'_\sigma(M_x) = \begin{cases} -k_y + (n - k_z + 1), & \sigma = (\pm - -), \\ (n - k_z + 1), & \sigma = (\pm + -), \\ k_y, & \sigma = (\pm + +), \\ 0, & \text{otherwise.} \end{cases} \quad (151)$$

Then using arguments similar to Proposition 3.2.1, one can show that

$$d_{G'_{x,M_x,\epsilon}}(\xi_\sigma^x) = W'_\sigma(M_x) \quad (152)$$

where  $\xi_\sigma^x$  is a regular value of  $G'_{x,M_x,\epsilon}$  and  $\xi_\sigma = \gamma_\sigma^{-1}(\xi_\sigma^x)$  satisfies (93).

Similarly for  $G''_{y,M_y,\epsilon}$ , we define the function  $g''_{m,\epsilon}$  on the quarter-annuli  $2\rho_{m-1} \leq \rho \leq \rho_m$  for  $1 \leq m \leq M_y$  as follows -

$$g''_{m,\epsilon}(w) = \begin{cases} \frac{\bar{w}}{\sqrt{\epsilon}\rho_m}, & 1 \leq m \leq 2(n - k_x - k_y + 1), m \text{ odd}, \\ \frac{\rho_{m-1}}{\sqrt{\epsilon}\bar{w}}, & 1 \leq m \leq 2(n - k_x - k_y + 1), m \text{ even}, \\ g_{m,\epsilon}(w), & 2(n - k_x - k_y + 1) < m \leq M_y. \end{cases} \quad (153)$$

Then  $\Gamma''_{M_y,\epsilon}$  and  $G''_{y,M_y,\epsilon}$  are defined by analogy with (111) and (120) respectively. For  $1 \leq m \leq 2(n - k_x - k_y + 1)$  and  $m$  odd,  $g''_{m,\epsilon}$  covers the pair of adjacent octants  $\Sigma_\sigma$ ,  $\sigma = (+ - \pm)$  exactly once with positive orientation and for  $m$  even,  $g''_{m,\epsilon}$  covers the pair of adjacent octants  $\Sigma_\sigma$ ,  $\sigma = (+ + \pm)$  exactly once with positive orientation. For  $2(n - k_x - k_y + 1) < m \leq M_y$ ,  $g''_{m,\epsilon}$  coincides with  $g_{m,\epsilon}$  defined in (92). One can directly check that  $G''_{y,M_y,\epsilon}$  satisfies the tangent boundary conditions on  $Q_{y,\epsilon} \cap \partial Q$  and has Dirichlet energy

$$E\left(G''_{y,M_y,\epsilon}\right) = \int_{Q_{y,\epsilon}} \mathcal{H}\left(G''_{y,M_y,\epsilon}\right) d^2w \leq 2\pi M_y + D\epsilon \quad (154)$$

where  $D$  is a positive constant independent of  $\epsilon$ . The algebraic degrees are readily computed to be

$$d_{G''_{y,M_y,\epsilon}}(\xi_\sigma^y) = W''_\sigma(M_y) \quad (155)$$

where  $\xi_\sigma^y$  is a regular value of  $G''_{y,M_y,\epsilon}$  and  $\xi_\sigma = \gamma_y^{-1}(\xi_\sigma^y)$  satisfies (93) and

$$W''_\sigma(M_y) = \begin{cases} k_x - 1 + (n - k_x - k_y + 1), & \sigma = (+\pm+), \\ (n - k_x - k_y + 1), & \sigma = (-\pm+), \\ -k_x, & \sigma = (-\pm-), \\ 0, & \text{otherwise.} \end{cases} \quad (156)$$

Given  $F_{H_0}$ ,  $G'_{x,M_x,\epsilon}$  and  $G''_{y,M_y,\epsilon}$ , the function  $K_{H_0,M,\epsilon}$  is defined as in (126).

**Case 2d:**  $k_z \leq n \leq k_x + k_z - 2$ : Take  $H_0$  and  $M = (M_x, M_y, M_z)$  as in Case 1d.

**Case 2e:**  $k_x + k_z - 1 \leq n \leq k_y + k_z - 2$ : Take  $H_0$  and  $M = (M_x, M_y, M_z)$  as in Case 1e.

**Case 2f:**  $k_y + k_z - 1 \leq n \leq \sum_j k_j - 2$ : Take  $H_0$  and  $M = (M_x, M_y, M_z)$  as in Case 1f.

### Remaining cases

This deals with cases where one or more of the  $k_j$ 's is either zero or negative. Let  $H$  be an arbitrary nonconformal homotopy class with  $e_j = +1$  for all  $j$  and  $k_j \leq 0$  for some  $j$ . As in cases 1 and 2, we can explicitly specify  $M = (M_x, M_y, M_z)$  and  $H_0$  in these cases so that the overall representative  $\nu$  is defined as in (126). We briefly outline the details here for completeness. We denote the set of wrapping numbers by  $\{w_\sigma\}$ . Then the octant  $\Sigma_{\sigma_+}$  with  $\sigma_+ = (\text{sgn } k_x, \text{sgn } k_y, \text{sgn } k_z)$  has the largest positive wrapping number and the octant  $\Sigma_{\sigma_-}$  with  $\sigma_- = (-\text{sgn } k_x, -\text{sgn } k_y, -\text{sgn } k_z)$  has the smallest negative wrapping number. (There may be more than one octant with largest positive wrapping number or smallest negative wrapping number but we adhere to these choices for definiteness.)

As before, we look at the triad of octants adjacent to  $\Sigma_{\sigma_+}$  and  $\Sigma_{\sigma_-}$ . We define  $M_j$  quarter-sphere configurations on each sub-domain  $Q_{j,\epsilon}$ . The conformal quarter-sphere configurations in  $Q_{x,\epsilon}$  cover  $\Sigma_{\sigma_-}$  and  $\Sigma_{(\text{sgn } k_x, -\text{sgn } k_y, -\text{sgn } k_z)}$  once with negative orientation (examples of which are the  $m$  odd case in (92)) whereas the anticonformal quarter-sphere configurations in  $Q_{x,\epsilon}$  cover  $\Sigma_{\sigma_+}$  and  $\Sigma_{(-\text{sgn } k_x, \text{sgn } k_y, \text{sgn } k_z)}$  with positive orientation (examples of which are the  $m$  even case in (92)). Similarly, the conformal quarter-sphere configurations in  $Q_{y,\epsilon}$  cover  $\Sigma_{\sigma_-}$  and  $\Sigma_{(-\text{sgn } k_x, \text{sgn } k_y, -\text{sgn } k_z)}$  once with negative orientation whereas the anticonformal quarter-sphere configurations in  $Q_{y,\epsilon}$  cover  $\Sigma_{\sigma_+}$  and  $\Sigma_{(\text{sgn } k_x, -\text{sgn } k_y, \text{sgn } k_z)}$  with positive orientation. Finally, the conformal quarter-sphere configurations in  $Q_{z,\epsilon}$  cover  $\Sigma_{\sigma_-}$  and  $\Sigma_{(-\text{sgn } k_x, -\text{sgn } k_y, \text{sgn } k_z)}$  once with negative orientation and the anticonformal quarter-sphere configurations cover  $\Sigma_{\sigma_+}$  and  $\Sigma_{(\text{sgn } k_x, \text{sgn } k_y, -\text{sgn } k_z)}$  once with positive orientation.

In each case, the algebraic degrees  $d_{G_{x,M_x,\epsilon}}(\sigma)$ ,  $d_{G_{y,M_y,\epsilon}}(\sigma)$  and  $d_{\Gamma_{M_z,\epsilon}}(\sigma)$  can be computed as in (124) and (113). Once the  $M_j$ 's are specified, we define the set of numbers  $\{w_{\sigma,0}\}$  as shown below

$$w_{\sigma,0} = w_\sigma - d_{G_{x,M_x,\epsilon}}(\sigma) - d_{G_{y,M_y,\epsilon}}(\sigma) - d_{\Gamma_{M_z,\epsilon}}(\sigma). \quad (157)$$

The  $\{w_{\sigma,0}\}$ 's constitute the set of wrapping numbers for a conformal or anticonformal bulk topology  $H_0$ . Given  $H_0$  and  $M = (M_x, M_y, M_z)$ , the representative  $\nu$  is defined as in (126).

**Lemma 3.3.1.** *For every nonconformal homotopy class  $H$  with wrapping numbers  $\{w_\sigma\}$ , we define the representative  $K_{H_0,M,\epsilon}$  in (126) where  $H_0$  and  $M$  are explicitly specified. Let  $\{w_{\sigma,0}\}$  denote the*

wrapping numbers of the homotopy class  $H_0$ . Then

$$\sum_{\sigma} |w_{\sigma,0}| + |d_{\Gamma_{M_z,\epsilon}}(\sigma)| + |d_{G_{x,M_x,\epsilon}}(\sigma)| + |d_{G_{y,M_y,\epsilon}}(\sigma)| = \sum_{\sigma} |w_{\sigma}| + \Delta(H) \quad (158)$$

where  $\Delta(H)$  has been defined in (10).

*Proof.* For each of the cases in Section 3.3, we explicitly specify  $H_0 = (e_0, k_0, \Omega_0)$  and a set of three non-negative integers  $M = (M_x, M_y, M_z)$  as shown above. Given  $H_0 = (e_0, k_0, \Omega_0)$ , the corresponding wrapping numbers  $\{w_{\sigma,0}\}$  can be computed using formula (5). Similarly, given  $M_j$ , the algebraic degrees  $d_{G_{x,M_x,\epsilon}}(\sigma)$ ,  $d_{G_{y,M_y,\epsilon}}(\sigma)$  and  $d_{\Gamma_{M_z,\epsilon}}(\sigma)$  are given in (124) and (113) respectively where we have dropped explicit reference to regular values, since these algebraic degrees only depend on the octant  $\Sigma_{\sigma}$  in question. One can directly substitute the values of  $w_{\sigma,0}$ ,  $d_{G_{x,M_x,\epsilon}}(\sigma)$ ,  $d_{G_{y,M_y,\epsilon}}(\sigma)$  and  $d_{\Gamma_{M_z,\epsilon}}(\sigma)$  and check that

$$\sum_{\sigma} |w_{\sigma,0}| + |d_{\Gamma_{M_z,\epsilon}}(\sigma)| + |d_{G_{x,M_x,\epsilon}}(\sigma)| + |d_{G_{y,M_y,\epsilon}}(\sigma)| = \sum_{\sigma} |w_{\sigma}| + \Delta(H) \quad (159)$$

in all cases, where  $\Delta(H)$  is defined in (10).

We outline the calculations for case 1a as an illustration. For case 1a, the bulk topology  $H_0$  is conformal with invariants as in (131). The corresponding wrapping numbers  $\{w_{\sigma,0}\}$  are

$$\begin{aligned} w_{+++ ,0} &= 0, \quad w_{-++ ,0} = 1 - k_x, \quad w_{--+ ,0} = n - k_x - k_y + 1, \quad w_{+-+ ,0} = n - k_y + 1 \\ w_{++- ,0} &= n - k_z + 1, \quad w_{-+- ,0} = n - k_x - k_z + 1, \quad w_{--- ,0} = 2n - \sum_j k_j + 1, \quad w_{+-- ,0} = 2n - k_y - k_z \end{aligned} \quad (160)$$

The algebraic degrees  $d_{\Gamma_{M_z,\epsilon}}(\sigma)$ ,  $d_{G_{x,M_x,\epsilon}}(\sigma)$  and  $d_{G_{y,M_y,\epsilon}}(\sigma)$  are given by (113) and (124) respectively i.e.

$$d_{G_{x,M_x,\epsilon}}(\sigma) = \begin{cases} n, & \sigma = (\pm ++), \\ -n, & \sigma = (\pm --), \\ 0, & \text{otherwise} \end{cases} \quad (161)$$

and  $d_{\Gamma_{M_z,\epsilon}}(\sigma) = 0$  and  $d_{G_{y,M_y,\epsilon}}(\sigma) = 0$  for all  $\sigma$  since  $M_z = M_y = 0$ .

We substitute these values into the left-hand side of (158) and obtain

$$\sum_{\sigma} |w_{\sigma,0}| + |d_{\Gamma_{M_z,\epsilon}}(\sigma)| + |d_{G_{x,M_x,\epsilon}}(\sigma)| + |d_{G_{y,M_y,\epsilon}}(\sigma)| = \sum_{\sigma} |w_{\sigma}| + 2 \min(n, k_x - 1). \quad (162)$$

Next, we compute  $\Delta(H)$  for all nonconformal homotopy classes within Case 1a i.e. with  $1 \leq n \leq k_y - 1$ . This case can be partitioned into two sub-cases according to whether  $n \leq k_x - 1$  or  $n \geq k_x - 1$ . If  $n \leq k_x - 1$ , then  $w_{+++}$  is the only positive wrapping number whereas if  $n \geq k_x - 1$ , then  $w_{+++}$  and  $w_{-++}$  are the only two non-negative wrapping numbers. The factor  $\chi$  in (10) vanishes by definition (see (9)) since  $k_j > 0$  for all  $j$ . Here  $\sigma_+ = (+++)$  and  $\sigma_- = (---)$  in the definition of  $\Delta(H)$  in (10). We note that

$$|w_{---}| - \sum_{\sigma \sim (---)} \Phi(-w_{\sigma}) = \sum_j k_j - n - 1 - \left( 2 \sum_j k_j - 3n - 3 \right) = 2n + 2 - \sum_j k_j < 0$$



for  $1 \leq n \leq k_y - 1$ , where  $\Phi(x) = \frac{1}{2}(x + |x|)$ . Therefore, we need only compute

$$|w_{+++}| - \sum_{\sigma \sim (+++)} \Phi(w_\sigma)$$

in (10).

For  $1 \leq n \leq k_x - 1$ ,

$$w_{+++} - \sum_{\sigma \sim (+++)} \Phi(w_\sigma) = n \quad (163)$$

whereas for  $k_x - 1 \leq n \leq k_y - 1$ ,

$$w_{+++} - \sum_{\sigma \sim (+++)} \Phi(w_\sigma) = k_x - 1. \quad (164)$$

Combining (163) and (164), we obtain

$$\Delta(H) = 2 \max \left( 0, w_{+++} - \sum_{\sigma \sim (+++)} \Phi(w_\sigma), |w_{---}| - \sum_{\sigma \sim (---)} \Phi(-w_\sigma) \right) = 2 \min(n, k_x - 1) \quad (165)$$

and a direct comparison with (162) establishes the required result.  $\square$

### 3.4 Proof of Theorem 2

*Proof.* Let  $H$  denote an arbitrary nonconformal homotopy class in  $\mathcal{C}_T(O; S^2)$  with associated wrapping numbers  $\{w_\sigma\}$ . For  $H$  conformal or anticonformal,  $\Delta(H) = 0$  by definition and we can construct a rational representative  $F_H$  of the form (86). As demonstrated in [11],  $E(F_H) = \pi \sum_\sigma |w_\sigma|$  (refer to (71)), consistent with the upper bound in Theorem 2.

For  $H$  nonconformal, we specify a conformal or anticonformal bulk topology  $H_0$  and a set of three non-negative integers  $M = (M_x, M_y, M_z)$ . The function  $K_{H_0, M, \epsilon}$  is defined as in (126) and the representative  $\nu = P^{-1}(K_{H_0, M, \epsilon})$ . One can readily verify that the function  $K_{H_0, M, \epsilon}$  belongs to the space  $\mathcal{C}_T(Q, \mathbb{C}^*)$ . To see why, it suffices to note that  $K_{H_0, M, \epsilon}$  is real on the real axis, imaginary on the imaginary axis and of unit modulus on the unit circle. This is immediate from the definitions of  $F_{H_0}$  and  $G_{j, M_j, \epsilon}$ . The interpolatory functions,  $((1-s)/G_{j, M_j, \epsilon} + s/F_{H_0})^{-1}$  and  $((1-s)G_{j, M_j, \epsilon} + sF_{H_0})$ , on  $Q_{j, 2\epsilon} - Q_{j, \epsilon}$ , satisfy the tangent boundary conditions from the definition of  $G_{j, M_j, \epsilon}$  in (120). Further, we continuously interpolate between the different quarter-sphere configurations and between  $F_{H_0}$  and  $G_{j, M_j, \epsilon}$ , and this ensures the continuity of the overall configuration  $K_{H_0, M, \epsilon}$ .

Let  $\{w_{\sigma, K}\}$  and  $\{w_{\sigma, 0}\}$  denote the wrapping numbers of  $K_{H_0, M, \epsilon}$  and  $F_{H_0}$  respectively. Then  $w_{\sigma, K}$  and  $w_{\sigma, 0}$  are related by

$$w_{\sigma, K} = w_{\sigma, 0} + d_{G_{x, M_x, \epsilon}}(\sigma) + d_{G_{y, M_y, \epsilon}}(\sigma) + d_{\Gamma_{M_z, \epsilon}}(\sigma) \quad (166)$$

where  $d_{\Gamma_{M_z, \epsilon}}(\sigma)$ ,  $d_{G_{x, M_x, \epsilon}}(\sigma)$  and  $d_{G_{y, M_y, \epsilon}}(\sigma)$  are given by (113) and (124) respectively (in the special case 2c, we replace  $d_{G_{x, M_x, \epsilon}}(\sigma)$  and  $d_{G_{y, M_y, \epsilon}}(\sigma)$  by  $d_{G'_{x, M_x, \epsilon}}$  and  $d_{G''_{y, M_y, \epsilon}}$  in (152) and (155) respectively). One can directly compute the  $w_{\sigma, K}$ 's from (166) and check that

$$w_{\sigma, K} = w_{\sigma} \quad \forall \sigma \quad (167)$$

so that  $\nu = P^{-1}(K_{H_0, M, \epsilon}) \in H$  as required.

The Dirichlet energy,  $E(K_{H_0, M, \epsilon})$ , is the sum of the energy contributions from the different sub-domains and

$$E(K_{H_0, M, \epsilon}) \leq \int \int_{Q_0} \mathcal{H}(F_{H_0}) d^2 w + \sum_j \int \int_{Q_{j, \epsilon}} \mathcal{H}(G_{j, M_j, \epsilon}) d^2 w + C\epsilon \quad (168)$$

where  $C$  is a positive constant independent of  $\epsilon$  (the energy of the interpolatory functions on  $Q_{j, 2\epsilon} \setminus Q_{j, \epsilon}$  has been absorbed into the  $C\epsilon$ -contribution on the right-hand side of (168) and  $G_{z, M_z, \epsilon} = \Gamma_{M_z, \epsilon}$  in (111)). The function  $K_{H_0, M, \epsilon}$  is either conformal or anticonformal everywhere by construction. Therefore, by using arguments similar to (71) and (87), we have that

$$\begin{aligned} \int \int_{Q_0} \mathcal{H}(F_{H_0}) d^2 w &\leq \pi \sum_{\sigma} |w_{\sigma, 0}| + C_1 \epsilon \\ \int \int_{Q_{j, \epsilon}} \mathcal{H}(G_{j, M_j, \epsilon}) d^2 w &\leq \pi \sum_{\sigma} |d_{G_{j, M_j, \epsilon}}(\sigma)| + C_2 \epsilon, \quad j = x, y, z \end{aligned} \quad (169)$$

where  $C_1$  and  $C_2$  are positive constants independent of  $\epsilon$ . We substitute (169) into (168) to get the upper bound

$$E(K_{H_0, M, \epsilon}) \leq \pi \sum_{\sigma} \left( |w_{\sigma, 0}| + |d_{\Gamma_{M_z, \epsilon}}(\sigma)| + |d_{G_{x, M_x, \epsilon}}(\sigma)| + |d_{G_{y, M_y, \epsilon}}(\sigma)| \right) + D\epsilon \quad (170)$$

for a positive constant  $D$  independent of  $\epsilon$ . Finally, from Lemma 3.3.1, we have that

$$\sum_{\sigma} \left( |w_{\sigma, 0}| + |d_{\Gamma_{M_z, \epsilon}}(\sigma)| + |d_{G_{x, M_x, \epsilon}}(\sigma)| + |d_{G_{y, M_y, \epsilon}}(\sigma)| \right) = \sum_{\sigma} |w_{\sigma}| + \Delta(H)$$

and substituting the above into (170) yields

$$E(K_{H_0, M, \epsilon}) \leq \pi \left( \sum_{\sigma} |w_{\sigma}| + \Delta(H) \right) + D\epsilon. \quad (171)$$

In the limit  $\epsilon \rightarrow 0$ , we recover the upper bound in Theorem 2.  $\square$

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## A Proof of Propositions 2.4.1 and 2.4.2

### A.1 Spelling length on words.

We state some basic definitions and notation concerning words and free groups. Let

$$\mathcal{A}_N = \{X_1, \dots, X_N, X_1^{-1}, \dots, X_N^{-1}\}. \quad (\text{A.1})$$

Elements of  $\mathcal{A}_N$  are called *letters*.  $X_r^{-1}$  is called the *inverse* of  $X_r$ , and vice versa. Sometimes we denote letters by  $A, B, C$ , etc. A *word of length  $k$*  on  $\mathcal{A}_N$  is a  $k$ -tuple of letters  $\mathbf{U} = (U(1), \dots, U(k))$ , where  $U(j) \in \mathcal{A}_N$ . The set of words of length  $k$  is denoted  $\mathcal{L}_N^k$ . The word of zero length is denoted  $\mathbf{e}$ , and we write  $\mathcal{L}_N^0 = \{\mathbf{e}\}$ . The set of all words is given by

$$\mathcal{L}_N = \cup_{k=0}^{\infty} \mathcal{L}_N^k. \quad (\text{A.2})$$

Let  $L$  denote the length function on  $\mathcal{L}_N$ , so that  $L(\mathbf{U}) = k$  for  $\mathbf{U} \in \mathcal{L}_N^k$ . Adjunction defines a product operation on  $\mathcal{L}_N$ ; given  $\mathbf{U} \in \mathcal{L}_N^k$  and  $\mathbf{V} \in \mathcal{L}_N^l$ , define  $(\mathbf{U}, \mathbf{V}) \in \mathcal{L}_N^{k+l}$  by

$$(\mathbf{U}, \mathbf{V}) = (U(1), \dots, U(k), V(1), \dots, V(l)). \quad (\text{A.3})$$

Then  $(\mathbf{U}, \mathbf{e}) = (\mathbf{e}, \mathbf{U}) = \mathbf{U}$ . Clearly,

$$L((\mathbf{U}, \mathbf{V})) = L(\mathbf{U}) + L(\mathbf{V}). \quad (\text{A.4})$$

We use the exponential notation  $X_r^j$  to denote the  $j$ -tuple  $(X_r, \dots, X_r)$  for  $j$  positive; for  $j$  negative,  $X_r^j$  denotes the  $j$ -tuple  $(X_r^{-1}, \dots, X_r^{-1})$ , and for  $j = 0$ , the identity  $\mathbf{e}$ . Also, for  $\mathbf{U} \in \mathcal{L}_N^k$  with  $k > 0$ , define  $\mathbf{U}^{-1} \in \mathcal{L}_N^k$  by

$$\mathbf{U}^{-1} = (U(k)^{-1}, \dots, U(1)^{-1}) \quad (\text{A.5})$$

and define  $\mathbf{e}^{-1}$  to be  $\mathbf{e}$ .

The *free group*  $F(X_1, \dots, X_N)$  is the set of equivalence classes in  $\mathcal{L}_N$  under all relations of the form

$$(\mathbf{U}, X_r, X_r^{-1}, \mathbf{V}) \sim (\mathbf{U}, X_r^{-1}, X_r, \mathbf{V}) \sim (\mathbf{U}, \mathbf{V}). \quad (\text{A.6})$$

Given  $\mathbf{U} \in \mathcal{L}_N$ , we denote its equivalence class in  $F(X_1, \dots, X_N)$  either by  $[\mathbf{U}]$  or by  $U$ . Multiplication in  $F(X_1, \dots, X_N)$  is defined and denoted by

$$UV = [(\mathbf{U}, \mathbf{V})] \quad (\text{A.7})$$

Inverses in  $F(X_1, \dots, X_N)$  are given by

$$U^{-1} = [\mathbf{U}^{-1}]. \quad (\text{A.8})$$

We introduce another length function on  $\mathcal{L}_N$ , denoted  $\lambda$ . As it will turn out to be equivalent to the spelling length  $\Lambda$  (cf Proposition A.1.1 below), we shall also refer to  $\lambda$  as the spelling length.  $\lambda$  is defined inductively as follows: On words of length 0, ie  $\mathbf{e}$ , we take

$$\lambda(\mathbf{e}) = 0. \quad (\text{A.9})$$

Given that  $\lambda$  is defined on words of length less than  $k$ , for  $\mathbf{U} \in \mathcal{L}_n^k$  we define

$$\lambda(\mathbf{U}) = \min \left( 1 + \lambda(\mathbf{U}_{2:k}), \min_{U(j)=U(1)^{-1}} \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:k}) \right), \quad (\text{A.10})$$

where  $\mathbf{U}_{a:b}$  is equal to  $(U(a), \dots, U(b))$  for  $b \geq a$  and to  $\mathbf{e}$  for  $b < a$ . From the definition (A.10), if  $\lambda(\mathbf{U}) < 1 + \lambda(\mathbf{U}_{2:k})$ , then there is an index  $j$  such that  $U(1)$  and  $U(j)$  are inverses, and  $\lambda(\mathbf{U}) = \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:k})$ . Proceeding recursively, we see that  $\lambda(\mathbf{U})$  is achieved by specifying a set of inverse-letter pairs

$$I = \{\{a_1, b_1\}, \dots, \{a_p, b_p\}\} \quad (\text{A.11})$$

such that

$$U(b_i) = U(a_i)^{-1} \text{ for all } \{a_i, b_i\} \in I, \quad (\text{A.12a})$$

$$a_i < a_j \text{ implies either } b_i < a_j \text{ or } b_i > b_j. \quad (\text{A.12b})$$

The last condition just means that the intervals  $[a_i, b_i]$  and  $[a_j, b_j]$  are either disjoint or else one contains the other as a proper subset.  $I$  is called a *pairing*. Given a word  $\mathbf{U}$  and a pairing  $I$ ,  $I$  is said to be *valid* for  $\mathbf{U}$  if (A.12) is satisfied. If  $i$  belongs to some pair in  $I$ , we say that  $i$  is *paired in*  $I$ . Paired letters do not contribute to the spelling length. Hence, if  $I$  is a valid pairing for  $\mathbf{U}$ ,

$$\lambda(\mathbf{U}) \leq L(\mathbf{U}) - 2|I|, \quad (\text{A.13})$$

where  $|I|$  is the number of elements (ie, pairs) in  $I$ . If  $\lambda(\mathbf{U}) = L(\mathbf{U}) - 2|I|$ , we say that  $I$  is an *optimal pairing*. The preceding discussion implies that every word has an optimal pairing.

$\lambda$  and  $\Lambda$  are equivalent in the following sense:

**Proposition A.1.1.**  *$\lambda$  descends to a function on the free group  $F(X_1, \dots, X_N)$  where it coincides with  $\Lambda$ . That is, if  $[\mathbf{U}] = [\mathbf{U}'] = U$ , then*

$$\lambda(\mathbf{U}) = \lambda(\mathbf{U}') = \Lambda(U).$$

The proof of Proposition A.1.1 makes use of several properties of  $\lambda$  which are established in the following lemmas.

**Lemma A.1.1** (Sub-additivity). *For all  $\mathbf{U}, \mathbf{V} \in \mathcal{L}_N$ ,*

$$\lambda((\mathbf{U}, \mathbf{V})) \leq \lambda(\mathbf{U}) + \lambda(\mathbf{V}).$$

*Proof.* By induction on  $L(\mathbf{U})$ . The statement is trivial for  $L(\mathbf{U}) = 0$ , ie  $\mathbf{U} = \mathbf{e}$ . Suppose it is true for all  $\mathbf{U}$  with  $L(\mathbf{U}) < k$ , and suppose  $L(\mathbf{U}) = k$ . From the definition (A.10),

$$\lambda((\mathbf{U}, \mathbf{V})) \leq \min \left( 1 + \lambda(\mathbf{U}_{2:k}, \mathbf{V}), \min_{U(j)=U(1)^{-1}} \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:k}, \mathbf{V}) \right) \quad (\text{A.14})$$

(we have omitted terms from indices  $j$  for which  $V(j) = U(1)^{-1}$  – hence we have an inequality rather than an equality in (A.14)). By the induction hypothesis,

$$\lambda(\mathbf{U}_{2:k}, \mathbf{V}) \leq \lambda(\mathbf{U}_{2:k}) + \lambda(\mathbf{V}), \quad \lambda(\mathbf{U}_{j+1:k}, \mathbf{V}) \leq \lambda(\mathbf{U}_{j+1:k}) + \lambda(\mathbf{V}). \quad (\text{A.15})$$

From (A.10), (A.14) and (A.15),

$$\lambda((\mathbf{U}, \mathbf{V})) \leq \min \left( 1 + \lambda(\mathbf{U}_{2:k}), \min_{U(j)=U(1)-1} \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:k}) \right) + \lambda(\mathbf{V}) = \lambda(\mathbf{U}) + \lambda(\mathbf{V}). \quad (\text{A.16})$$

□

**Lemma A.1.2** (Cyclicity). *Let  $\mathbf{U} = (U(1), \dots, U(k))$ . Then*

$$\lambda((U(k), U(1) \dots U(k-1))) = \lambda(\mathbf{U}).$$

*Proof.* By induction on  $L(\mathbf{U})$ . The statement is trivial for  $L(\mathbf{U}) = 0$  and  $L(\mathbf{U}) = 1$ . Given  $k > 1$ , suppose it is true for all  $\mathbf{U}$  with  $L(\mathbf{U}) < k$  and let  $L(\mathbf{U}) = k$ . Let  $\mathbf{U}' = (U(k), U(1), \dots, U(k-1))$ .

Let us compute  $\lambda(\mathbf{U})$ , applying the definition (A.10) twice, as follows: The first application gives

$$\lambda(\mathbf{U}) = \min \left( 1 + \lambda(\mathbf{U}_{2:k}), \min_{U(i)=U(1)-1} \lambda(\mathbf{U}_{2:i-1}) + \lambda(\mathbf{U}_{i+1:k}) \right). \quad (\text{A.17})$$

The induction hypothesis implies that

$$\lambda(\mathbf{U}_{2:k}) = \lambda((U(k), \mathbf{U}_{2:k-1})), \quad \lambda(\mathbf{U}_{i+1:k}) = \lambda((U(k), \mathbf{U}_{i+1:k-1})). \quad (\text{A.18})$$

Substituting into (A.17) and applying (A.10) again to terms in which  $U(k)$  appears as the first letter, we get that

$$\begin{aligned} \lambda(\mathbf{U}) = \min & \left( 2 + \lambda(\mathbf{U}_{2:k-1}), 1 + \min_{U(j)=U(k)-1} \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:k-1}), \right. \\ & 1 + \min_{U(i)=U(1)-1} \lambda(\mathbf{U}_{2:i-1}) + \lambda(\mathbf{U}_{j+1:k-1}), \\ & \left. \min_{U(i)=U(1)-1} \min_{\substack{U(j)=U(k)-1 \\ j>i+1}} \lambda(\mathbf{U}_{2:i-1}) + \lambda(\mathbf{U}_{i+1:j-1}) + \lambda(\mathbf{U}_{j+1:k-1}) \right). \quad (\text{A.19}) \end{aligned}$$

Next we compute  $\lambda(\mathbf{U}')$ , applying the definition (A.10) twice, as follows: The first application gives

$$\lambda(\mathbf{U}') = \min \left( 1 + \lambda(\mathbf{U}_{1:k-1}), \min_{U(i)=U(k)-1} \lambda(\mathbf{U}_{1:i-1}) + \lambda(\mathbf{U}_{i+1:k-1}) \right). \quad (\text{A.20})$$

Applying (A.10) to terms in which the first letter of the argument of  $\lambda$  is  $U(1)$ , we get

$$\begin{aligned} \lambda(\mathbf{U}') = \min & \left( 2 + \lambda(\mathbf{U}_{2:k-1}), 1 + \min_{U(j)=U(1)-1} \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:k-1}), \right. \\ & 1 + \min_{U(i)=U(k)-1} \lambda(\mathbf{U}_{2:i-1}) + \lambda(\mathbf{U}_{i+1:k-1}), \\ & \left. \min_{U(i)=U(k)-1} \min_{\substack{U(j)=U(1)-1 \\ j<i-1}} \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:i-1}) + \lambda(\mathbf{U}_{i+1:k-1}) \right). \quad (\text{A.21}) \end{aligned}$$

Comparison of (A.19) and (A.21) shows that  $\lambda(\mathbf{U}) = \lambda(\mathbf{U}')$ . □

**Lemma A.1.3** (Zero length words).

$$\lambda(\mathbf{U}) = 0 \text{ if and only if } U = e.$$

*Proof.* First, we suppose that  $\lambda(\mathbf{U}) = 0$ . We proceed by induction on  $L(\mathbf{U})$ . The assertion is true for  $L(\mathbf{U}) = 0$ , by definition. Suppose it is true for all  $\mathbf{U}$  of length less than  $k$ , and let  $\mathbf{U}$  be a word of length  $k$  with  $\lambda(\mathbf{U}) = 0$ . Then there is some  $j$  with  $1 < j \leq k$  such that  $U(j) = U(1)^{-1}$  and

$$0 = \lambda(\mathbf{U}) = \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:k}). \quad (\text{A.22})$$

Since  $\lambda$  is nonnegative, it follows that  $\lambda(\mathbf{U}_{2:j-1}) = \lambda(\mathbf{U}_{j+1:k}) = 0$ . By the induction hypothesis, it follows that  $[\mathbf{U}_{2:j-1}] = [\mathbf{U}_{j+1:k}] = e$ . Therefore,

$$U = U(1) [\mathbf{U}_{2:j-1}] U(j) [\mathbf{U}_{j+1:k}] = U(1)U(j) = e. \quad (\text{A.23})$$

Next, suppose that  $U = e$ . Then  $L(\mathbf{U})$  is even and

$$\mathbf{U} = (X_{r_1}, \dots, X_{r_m}, X_{r_m}^{-1}, \dots, X_{r_1}^{-1}). \quad (\text{A.24})$$

It follows that  $I = \{\{1, 2m\}, \{2, 2m-1\}, \dots, \{m, m+1\}\}$  is a valid pairing for  $\mathbf{U}$  and that  $L(\mathbf{U}) - 2|I| = 2m - 2m = 0$ . From (A.13) and the fact that  $\lambda$  is nonnegative, it follows that  $\lambda(\mathbf{U}) = 0$ .  $\square$

**Lemma A.1.4.** If  $\mathbf{h} \in \mathcal{L}_N$  and  $X \in \mathcal{A}_N$ , then

$$\lambda((\mathbf{h}, X, \mathbf{h}^{-1})) = 1.$$

*Proof.* Let  $L(\mathbf{h}) = k$ . Then  $I = \{\{1, 2k+1\}, \{2, 2k\}, \dots, \{k, k+2\}\}$  is a valid pairing for  $(\mathbf{h}, X, \mathbf{h}^{-1})$ , so that, from (A.13),

$$\lambda(\mathbf{h}, X, \mathbf{h}^{-1}) \leq L(\mathbf{h}, X, \mathbf{h}^{-1}) - 2|I| = 2k + 1 - 2k = 1. \quad (\text{A.25})$$

On the other hand, since  $[(\mathbf{h}, X, \mathbf{h}^{-1})] \neq e$ , it follows from Lemma A.1.3 that  $\lambda(\mathbf{h}, X, \mathbf{h}^{-1}) > 0$ . Therefore, we may conclude that  $\lambda(\mathbf{h}, X, \mathbf{h}^{-1}) = 1$ .  $\square$

We proceed to the proof of Proposition A.1.1.

*Proof of Proposition A.1.1.* First, we show that  $\lambda(\mathbf{U}) = \lambda(\mathbf{U}')$  for  $[\mathbf{U}] = [\mathbf{U}']$ . In view of the defining relations (A.6) for  $F(X_1, \dots, X_N)$ , it suffices to show that

$$\lambda((\mathbf{U}, X^{-1}, X, \mathbf{V})) = \lambda((\mathbf{U}, X, X^{-1}, \mathbf{V})) = \lambda((\mathbf{U}, \mathbf{V})). \quad (\text{A.26})$$

We will just consider  $\lambda((\mathbf{U}, X^{-1}, X, \mathbf{V}))$ ; the argument for  $\lambda((\mathbf{U}, X, X^{-1}, \mathbf{V}))$  is similar.

By cyclicity (Lemma A.1.2), it suffices to show that for  $\mathbf{W} \in \mathcal{L}_N^k$

$$\lambda((X^{-1}, X, \mathbf{W})) = \lambda(\mathbf{W}). \quad (\text{A.27})$$

From (A.10),

$$\lambda((X^{-1}, X, \mathbf{W})) = \min \left( 1 + \lambda((X, \mathbf{W})), \lambda(\mathbf{W}), \min_{W(j)=X} \lambda((X, \mathbf{W}_{1:j-1})) + \lambda(\mathbf{W}_{j+1:k}) \right). \quad (\text{A.28})$$

To establish (A.27), we show that the first and third members of the right-hand side of (A.28) are not smaller than the second, namely  $\lambda(\mathbf{W})$ . We start with the first member, namely  $1 + \lambda((X, \mathbf{W}))$ . Applying (A.10), we get that

$$1 + \lambda((X, \mathbf{W})) = \min \left( 2 + \lambda(\mathbf{W}), \min_{W(j)=X^{-1}} 1 + \lambda(\mathbf{W}_{1:j-1}) + \lambda(\mathbf{W}_{j+1:k}) \right). \quad (\text{A.29})$$

By subadditivity (Lemma A.1.1),

$$1 + \lambda(\mathbf{W}_{1:j-1}) + \lambda(\mathbf{W}_{j+1:k}) \geq \lambda(\mathbf{W}). \quad (\text{A.30})$$

Therefore, from (A.29) and (A.30),

$$1 + \lambda((X, \mathbf{W})) \geq \lambda(\mathbf{W}). \quad (\text{A.31})$$

Referring to the third member of the right-hand side of (A.28), we have, for  $W(j) = X$ , that  $\lambda((X, \mathbf{W}_{1:j-1})) = \lambda(\mathbf{W}_{1:j})$  (cyclicity again), so that, by subadditivity,

$$\lambda((X, \mathbf{W}_{1:j-1})) + \lambda(\mathbf{W}_{j+1:k}) = \lambda(\mathbf{W}_{1:j}) + \lambda(\mathbf{W}_{j+1:k}) \geq \lambda(\mathbf{W}), \quad (\text{A.32})$$

as required.

Next, we show that  $\lambda(\mathbf{U}) = \Lambda(U)$ . We proceed by induction. For words of length zero, this follows from Lemma A.1.3. Suppose the statement is true for words of length less than  $k$ , and let  $\mathbf{U}$  have length  $k$ . Let  $n = \Lambda(U)$ . Then  $U$  has a spelling of length  $n$ , ie

$$U = h_1 X_{r_1} h_1^{-1} \cdots h_n X_{r_n} h_n^{-1} \quad (\text{A.33})$$

for some  $h_i \in F(X_1, \dots, X_N)$  and  $X_{r_i} \in \mathcal{A}_N$ . Let  $\mathbf{h}_i$  be words corresponding to  $h_i$ . Then

$$\lambda(\mathbf{U}) = \lambda((\mathbf{h}_1, X_{r_1}, \mathbf{h}_1^{-1}), \dots, (\mathbf{h}_n, X_{r_n}, \mathbf{h}_n^{-1})). \quad (\text{A.34})$$

From subadditivity (Lemma A.1.1) and Lemma A.1.4, it follows that

$$\lambda(\mathbf{U}) \leq \lambda((\mathbf{h}_1, X_{r_1}, \mathbf{h}_1^{-1})) + \cdots + \lambda((\mathbf{h}_n, X_{r_n}, \mathbf{h}_n^{-1})) = n = \Lambda(U). \quad (\text{A.35})$$

It remains to show that  $\lambda(\mathbf{U}) \geq \Lambda(U)$ . From (A.10), we have either that

$$\lambda(\mathbf{U}) = 1 + \lambda(\mathbf{U}_{2:k}) \quad (\text{A.36})$$

or that

$$\lambda(\mathbf{U}) = \lambda(\mathbf{U}_{2:j-1}) + \lambda(\mathbf{U}_{j+1:k}) \quad (\text{A.37})$$

for some  $j$  with  $U(j) = U(1)^{-1}$ . In case (A.36) holds, use the induction hypothesis to conclude that

$$\lambda(\mathbf{U}) = 1 + \Lambda([\mathbf{U}_{2:k}]) \geq \Lambda(U(1)[\mathbf{U}_{2:k}]) = \Lambda(U), \quad (\text{A.38})$$

where we have used the fact (easily verified) that  $\Lambda$  is subadditive, ie  $\Lambda(UV) \leq \Lambda(U) + \Lambda(V)$ . On the other hand, if (A.37) holds, then the induction hypothesis implies that

$$\begin{aligned} \lambda(\mathbf{U}) &= \Lambda([\mathbf{U}_{2:j-1}]) + \Lambda([\mathbf{U}_{j+1:k}]) = \Lambda(U(1)[\mathbf{U}_{2:j-1}]U(1)^{-1}) + \Lambda([\mathbf{U}_{j+1:k}]) \\ &= \Lambda([\mathbf{U}_{1:j}]) + \Lambda([\mathbf{U}_{j+1:k}]) \geq \Lambda(U), \end{aligned} \quad (\text{A.39})$$

where in the second equation we have used the invariance of  $\Lambda$  under conjugation (easily verified) and in the third the subadditivity of  $\Lambda$ .  $\square$

## A.2 Set products of conjugacy classes of words

Let

$$S = \{U_1, \dots, U_q \mid U_j \in F(X_1, \dots, X_N)\} \quad (\text{A.40})$$

denote a set of elements of the free group, and let

$$\mathcal{V}(S) = \langle U_1 \rangle \cdots \langle U_q \rangle \quad (\text{A.41})$$

denote the set product of their conjugacy classes. We wish to determine the minimum of the spelling length over  $\mathcal{V}(S)$ . In view of Proposition A.1.1, we can work with words rather than elements of  $F(X_1, \dots, X_N)$ . Choose words  $\mathbf{U}_j$  so that  $[\mathbf{U}_j] = U_j$ , and let

$$\mathcal{V}(\mathbf{S}) = \{(\mathbf{h}_1, \mathbf{U}_1, \mathbf{h}_1^{-1}, \dots, \mathbf{h}_q, \mathbf{U}_q, \mathbf{h}_q^{-1}) \mid \mathbf{h}_j \in \mathcal{L}_N\}. \quad (\text{A.42})$$

Thus, every  $U \in \mathcal{V}(S)$  has a representative  $\mathbf{U} \in \mathcal{V}(\mathbf{S})$  with  $[\mathbf{U}] = U$ , and

$$\min_{U \in \mathcal{V}(S)} \Lambda(U) = \min_{\mathbf{U} \in \mathcal{V}(\mathbf{S})} \lambda(\mathbf{U}). \quad (\text{A.43})$$

Let

$$\mathbf{U} = (\mathbf{h}_1, \mathbf{U}_1, \mathbf{h}_1^{-1}, \dots, \mathbf{h}_q, \mathbf{U}_q, \mathbf{h}_q^{-1}) \in \mathcal{V}(\mathbf{S}), \quad (\text{A.44})$$

and suppose that  $L(\mathbf{U}) = k$ . Let  $\mathcal{I}(\mathbf{U}) = \{1, \dots, k\}$  denote the indices of the letters in  $\mathbf{U}$ . Let  $\mathcal{W}(\mathbf{U})$  denote the set of indices of the letters in the  $\mathbf{U}_j$ 's, and  $\mathcal{C}(\mathbf{U})$  the set of indices of the letters in the  $\mathbf{h}_j$ 's and  $\mathbf{h}_j^{-1}$ 's, so that

$$\mathcal{W}(\mathbf{U}) \cup \mathcal{C}(\mathbf{U}) = \mathcal{I}(\mathbf{U}). \quad (\text{A.45})$$

Indices in  $\mathcal{C}(\mathbf{U})$  naturally fall into pairs associated to conjugate letters in  $\mathbf{h}_j$  and  $\mathbf{h}_j^{-1}$ . For example, if  $\mathbf{h}_j = (A, B^{-1}, C)$ , then  $\mathbf{h}_j^{-1} = (C^{-1}, B, A^{-1})$ , and we say that the indices of  $A$  and  $A^{-1}$  are conjugate, as are the indices of  $B$  and  $B^{-1}$  and of  $C$  and  $C^{-1}$ . In general, for  $c \in \mathcal{C}(\mathbf{U})$ , let  $\bar{c} \in \mathcal{C}(\mathbf{U})$  denote the index to which it is conjugate. One can compute an explicit formula for  $\bar{c}$  but we won't be needing explicit formulae for this discussion.



Let  $I$  be a valid pairing for  $\mathbf{U}$ . We say that  $i, j \in \mathcal{W}(\mathbf{U})$  are *linked in  $I$*  if there exists a sequence of indices  $c_1, \dots, c_m$  in  $\mathcal{C}(\mathbf{U})$  such that

$$\{i, c_1\}, \{\bar{c}_1, c_2\}, \dots, \{\bar{c}_{m-1}, c_m\}, \{c_m, j\} \in I. \quad (\text{A.46})$$

We call  $c_1, \dots, c_m$  a *linking sequence*. If  $i$  and  $j$  are linked, the linking sequence between them is unique. It is clear that, if  $i$  and  $j$  are linked, then  $U(j) = U(i)^{-1}$ . Moreover, the  $U(c_r)$ 's are all equal to  $U(i)^{-1}$ , while the  $U(\bar{c}_r)$ 's are all equal to  $U(i)$ . We say that a pairing  $I$  is *reduced* if every  $i \in \mathcal{W}(\mathbf{U})$  is either unpaired or else is linked to some  $j \in \mathcal{W}(\mathbf{U})$ .

We next describe a procedure for removing letters from a word. Let  $\mathbf{U} \in \mathcal{L}_N$  be a word of length  $k$ , and let  $R$  be a subset of  $\mathcal{I}(\mathbf{U})$ . Define the *re-indexing map*  $\psi_R$  to be the bijection between  $\{1, \dots, k\} - R$  and  $\{1, \dots, k - |R|\}$  given by

$$\psi_R(i) = i - |\{j \in R \mid j < i\}| \quad (\text{A.47})$$

(ie,  $\psi_R(i)$  is  $i$  minus the number of indices in  $R$  less than  $i$ ). Define  $\mathbf{V} \in \mathcal{L}_N$  to be the word of length  $k - |R|$  given by

$$\mathbf{V} = (U(\psi_R^{-1}(1)), \dots, U(\psi_R^{-1}(k - |R|))) \quad (\text{A.48})$$

(so  $\mathbf{V}$  is just  $\mathbf{U}$  without the letters indexed by  $R$ ). We say that  $\mathbf{V}$  is the word obtained by *removing  $R$  from  $\mathbf{U}$* . Note that we may regard  $\psi_R$  as a bijection between  $\mathcal{I}(\mathbf{U}) - R$  and  $\mathcal{I}(\mathbf{V})$ . If  $\mathbf{U}$  belongs to  $\mathcal{V}(\mathbf{S})$ , then, in general,  $\mathbf{V}$  does not. However, if  $R \subset \mathcal{C}(\mathbf{U})$  and  $c \in R$  implies that  $\bar{c} \in R$ , then removing  $R$  amounts to replacing one or more the  $\mathbf{h}_j$ 's by shorter words, and  $\mathbf{V}$  belongs to  $\mathcal{V}(\mathbf{S})$  as well.

Given  $I$ , a valid pairing for  $\mathbf{U}$ , let us define a pairing  $J$  by

$$J = \{\{\psi_R(i), \psi_R(j)\} \mid i, j \in \mathcal{I}(\mathbf{U}) - R \text{ and } \{i, j\} \in I\}. \quad (\text{A.49})$$

That is,  $J$  contains all the re-indexed pairs of indices in  $I$  which haven't been removed from  $\mathbf{U}$ . It is straightforward to check that  $J$  is valid for  $\mathbf{V}$  (cf (A.12)). However,  $I$  optimal does not imply that  $J$  is optimal, nor does  $I$  reduced imply that  $J$  is reduced. We say that  $J$  is the pairing obtained from *removing  $R$  from  $I$* .

The following proposition shows that the minimum spelling length on  $\mathcal{V}(\mathbf{S})$  can be realised by a word with an optimal reduced pairing.

**Proposition A.2.1.** *Let  $m = \min_{U \in \mathcal{V}(\mathbf{S})} \Lambda(U)$ . Then there exists  $\mathbf{U} \in \mathcal{V}(\mathbf{S})$  with optimal reduced pairing  $I$  such that  $\lambda(\mathbf{U}) = m$ .*

*Proof.* Choose  $\mathbf{U}' \in \mathcal{V}(\mathbf{S})$  such that  $\lambda(\mathbf{U}') = m$ . Let  $I'$  be an optimal pairing for  $\mathbf{U}'$  (as discussed in Section A.1, such an optimal pairing exists). Let  $t$  denote the number of indices in  $\mathcal{W}(\mathbf{U}')$  which are paired in  $I'$  but which are not linked to an index in  $\mathcal{W}(\mathbf{U}')$ . If  $t = 0$ , then  $I'$  is reduced, and we are done. In what follows, we obtain a word  $\mathbf{U} \in \mathcal{V}(\mathbf{S})$  with  $\lambda(\mathbf{U}) = m$  that has  $(t - 1)$  paired but unlinked indices in  $\mathcal{W}(\mathbf{U})$ . Let  $i \in \mathcal{W}(\mathbf{U}')$  be an index which is paired in  $I'$  but which is not linked to an index in  $\mathcal{W}(\mathbf{U}')$ . Then  $i$  is paired with some (unique)  $c_1 \in \mathcal{C}(\mathbf{U}')$ . Either  $\bar{c}_1$  is unpaired, or else  $\bar{c}_1$  is paired to some (unique)  $c_2 \in \mathcal{C}(\mathbf{U}')$ . Continuing in this way, we produce a sequence of indices  $c_1, \dots, c_d \in \mathcal{C}$ , where  $\{i, c_1\}, \{\bar{c}_1, c_2\}, \dots, \{\bar{c}_{d-1}, c_d\}$  are paired in  $I'$  and  $\bar{c}_d$  is unpaired.

We remove the set of indices  $R = \{c_1, \bar{c}_1, \dots, c_d, \bar{c}_d\}$  from  $\mathbf{U}'$  and  $I'$  to obtain  $\mathbf{U}$  with valid pairing  $I$ . Since  $R \subset \mathcal{C}(\mathbf{U}')$  and the elements of  $R$  occur in conjugate pairs, we have that  $\mathbf{U} \in \mathcal{V}(\mathbf{S})$ . It is clear that  $L(\mathbf{U}) = L(\mathbf{U}') - 2d$  and  $|I| = |I'| - d$ . From (A.13), it follows that

$$\lambda(\mathbf{U}) \leq L(\mathbf{U}) - 2|I| = L(\mathbf{U}') - 2|I'| = \lambda(\mathbf{U}') = m, \quad (\text{A.50})$$

where the second-to-last equality holds because  $I'$  is, by assumption, optimal. On the other hand, Proposition A.1.1 implies that  $\lambda(\mathbf{U}) \geq m$ . Therefore,  $\lambda(\mathbf{U}) = m$ . By construction,  $\mathbf{U}$  has  $t - 1$  indices in  $\mathcal{W}(\mathbf{U})$  which are paired in  $I$  but not linked to indices in  $\mathcal{W}(\mathbf{U})$  (in particular, while  $i \in \mathcal{W}(\mathbf{U}')$  is such an index,  $\psi_R(i) \in \mathcal{W}(\mathbf{U})$  is, by construction, unpaired, and therefore is not). One can repeat the construction  $t$  times to get a word in  $\mathbf{S}$ , with optimal reduced pairing, that achieves the minimum spelling length.  $\square$

Let  $\mathbf{U} \in \mathcal{V}(\mathbf{S})$  and let  $I$  be an optimal reduced pairing for  $\mathbf{U}$ . Let  $[I]_W$  denote the number of indices in  $\mathcal{W}(\mathbf{U})$  which are paired in  $I$ , and  $[I]_C$  the number of indices in  $\mathcal{C}(\mathbf{U})$  which are paired in  $I$ , so that

$$2|I| = [I]_W + [I]_C. \quad (\text{A.51})$$

We have the following formula for the spelling length:

**Proposition A.2.2.** *Let  $m = \min_{U \in \mathcal{V}(\mathbf{S})} \Lambda(U)$ . Choose  $\mathbf{U} \in \mathcal{V}(\mathbf{S})$  with optimal reduced pairing  $I$  such that  $\lambda(\mathbf{U}) = m$  (such a  $\mathbf{U}$  and  $I$  exist by Proposition A.2.1). Then every index in  $\mathcal{C}(\mathbf{U})$  is paired in  $I$ , and*

$$\lambda(\mathbf{U}) = \sum_{r=1}^q L(\mathbf{U}_r) - [I]_W. \quad (\text{A.52})$$

*Proof.* Suppose  $c_1 \in \mathcal{C}(\mathbf{U})$  is not paired. Then either  $\bar{c}_1$  is unpaired, or else it is paired to some  $c_2 \in \mathcal{C}(\mathbf{U})$ ; note that, since  $I$  is reduced,  $c_2$  cannot belong to  $\mathcal{W}(\mathbf{U})$ . Continue in this way to generate a sequence  $c_1, \dots, c_d$ , where  $c_1$  and  $\bar{c}_d$  are unpaired while  $\bar{c}_{r-1}$  is paired to  $c_r$  for  $1 < r \leq d$ . Remove the set of indices  $R = \{c_1, \bar{c}_1, \dots, c_d, \bar{c}_d\}$  from  $\mathbf{U}$  and  $I$  to obtain  $\mathbf{V} \in \mathcal{V}(\mathbf{S})$  with valid pairing  $J$ . Then  $L(\mathbf{V}) = L(\mathbf{U}) - 2d$  and  $|J| = |I| - (d - 1)$ . From (A.13), it follows that

$$\lambda(\mathbf{V}) \leq L(\mathbf{V}) - 2|J| = L(\mathbf{U}) - 2|I| - 2 = \lambda(\mathbf{U}) - 2 < m, \quad (\text{A.53})$$

in contradiction to the fact that  $\lambda(\mathbf{V}) \geq m$  for all  $\mathbf{V} \in \mathcal{V}(\mathbf{S})$ . It follows that all indices in  $\mathcal{C}(\mathbf{U})$  are paired. Therefore

$$2|I| = [I]_W + [I]_C = [I]_W + |\mathcal{C}(\mathbf{U})| = [I]_W + L(\mathbf{U}) - \sum_{r=1}^q L(\mathbf{U}_r). \quad (\text{A.54})$$

Since  $I$  is optimal,

$$\lambda(\mathbf{U}) = L(\mathbf{U}) - 2|I| = \sum_{r=1}^q L(\mathbf{U}_r) - [I]_W. \quad (\text{A.55})$$

$\square$

### A.3 Proof of Propositions 2.4.1 and 2.4.2

As the proofs of Propositions 2.4.1 and 2.4.2 are similar, we give details only for Proposition 2.4.1. We then briefly explain the differences that arise in the proof of Proposition 2.4.2.

*Proof of Proposition 2.4.1.* Let  $\mathcal{P}_{\mathbf{n},\mathbf{p}}$  denote the set of words

$$\begin{aligned} \mathcal{P}_{\mathbf{n},\mathbf{p}} = \{ & (\mathbf{h}_0, A^i, B^j, C^k, \mathbf{h}_0^{-1}, \\ & \mathbf{h}_1, A^{-1}, B^{-1}, C^{-1}, \mathbf{h}_1^{-1}, \dots, \mathbf{h}_n, A^{-1}, B^{-1}, C^{-1}, \mathbf{h}_n^{-1}, \\ & \mathbf{h}_{n+1}, C, B, A, \mathbf{h}_{n+1}^{-1}, \dots, \mathbf{h}_{n+p}, C, B, A, \mathbf{h}_{n+p}^{-1}), \quad \mathbf{h}_t \in \mathcal{L}_3 \}. \end{aligned} \quad (\text{A.56})$$

Then  $\mathbf{U} \in \mathcal{P}_{\mathbf{n},\mathbf{p}}$  implies that  $U \in \mathcal{P}_{n,p}$ . From Proposition A.1.1 it follows that

$$\min_{U \in \mathcal{P}_{n,p}} \Lambda(U) = \min_{\mathbf{U} \in \mathcal{P}_{\mathbf{n},\mathbf{p}}} \lambda(\mathbf{U}). \quad (\text{A.57})$$

For  $\mathbf{U} \in \mathcal{P}_{\mathbf{n},\mathbf{p}}$ , let

$$D_{n,p}(\mathbf{U}) = \lambda(\mathbf{U}) - (i + j + k - (n + p)). \quad (\text{A.58})$$

Then Proposition 2.4.1 is equivalent to showing that

$$D_{n,p}(\mathbf{U}) \geq 0. \quad (\text{A.59})$$

We refer to the three-tuples  $(A^{-1}, B^{-1}, C^{-1})$  in (A.56) as *negative triples*, and the three-tuples  $(C, B, A)$  as *positive triples*. Let us partition  $\mathcal{W}(\mathbf{U})$  into three sets as follows: Let  $\mathcal{W}_0(\mathbf{U})$  denote the set of indices of the letters of  $A^i B^j C^k$ ,  $\mathcal{W}_-(\mathbf{U})$  the set of indices of the negative triples, and  $\mathcal{W}_+(\mathbf{U})$  the set of indices of the positive triples. Then

$$\mathcal{W}(\mathbf{U}) = \mathcal{W}_0(\mathbf{U}) \cup \mathcal{W}_-(\mathbf{U}) \cup \mathcal{W}_+(\mathbf{U}). \quad (\text{A.60})$$

We proceed by assuming there exists  $\mathbf{U} \in \mathcal{P}_{\mathbf{n},\mathbf{p}}$  with  $D_{n,p}(\mathbf{U}) < 0$  and then deriving a contradiction. Lemma A.3.1 below, whose proof we give at the end of this section, shows that if there are unpaired indices in  $\mathcal{W}_-(\mathbf{U})$ , then we can construct  $\mathbf{V} \in \mathcal{P}_{\mathbf{n}-1,\mathbf{p}}$  with  $D_{n-1,p}(\mathbf{V}) < 0$ ; similarly, if there are unpaired indices in  $\mathcal{W}_+(\mathbf{U})$ , we can construct  $\mathbf{V} \in \mathcal{P}_{\mathbf{n},\mathbf{p}-1}$  with  $D_{n,p-1}(\mathbf{V}) < 0$ .

**Lemma A.3.1.** *Suppose there exists  $\mathbf{U} \in \mathcal{P}_{\mathbf{n},\mathbf{p}}$  with  $D_{n,p}(\mathbf{U}) < 0$ . Let  $I$  be an optimal reduced pairing for  $\mathbf{U}$  (which we may assume to exist by virtue of Proposition A.2.1). If  $\mathcal{W}_-(\mathbf{U})$  contains indices which are unpaired in  $I$ , then there exists  $\mathbf{V} \in \mathcal{P}_{\mathbf{n}-1,\mathbf{p}}$  such that  $D_{n-1,p}(\mathbf{V}) < 0$ . Similarly, if  $\mathcal{W}_+(\mathbf{U})$  contains indices which are unpaired in  $I$ , then there exists  $\mathbf{V} \in \mathcal{P}_{\mathbf{n},\mathbf{p}-1}$  such that  $D_{n,p-1}(\mathbf{V}) < 0$ .*

We apply Lemma A.3.1 repeatedly, reducing  $n$  or  $p$  by one each time, as many times as possible. We obtain thereby a word  $\mathbf{V}' \in \mathcal{P}_{\mathbf{m},\mathbf{q}}$ , where  $0 \leq m \leq n$  and  $0 \leq q \leq p$ , such that  $D_{m,q}(\mathbf{V}') < 0$ . We may assume that

$$m > 2q, \quad (\text{A.61})$$

since (60) implies that  $D_{m,q}(\mathbf{V}') \geq 4q - 2m$ . By Proposition A.2.1, there exists a word  $\mathbf{V} \in \mathcal{P}_{\mathbf{m},\mathbf{q}}$  with optimal reduced pairing  $J$  such that  $\lambda(\mathbf{V}) = \min_{\mathbf{U}' \in \mathcal{P}_{\mathbf{m},\mathbf{q}}} \lambda(\mathbf{U}')$ . Then  $\lambda(\mathbf{V}) \leq \lambda(\mathbf{V}')$ , so that

$D_{m,q}(\mathbf{V}) < 0$ . We may assume that every index in  $\mathcal{W}_\pm(\mathbf{V})$  is paired in  $J$ , as otherwise we could apply Lemma A.3.1 again. By Proposition A.2.2, every index in  $\mathcal{C}(\mathbf{V})$  is paired. Therefore, the only unpaired indices in  $\mathcal{I}(\mathbf{V})$  belong to  $\mathcal{W}_0(\mathbf{V})$ .

Let us remove the set of unpaired indices from  $\mathbf{V}$  and  $J$  to obtain  $\mathbf{W}$  and  $K$ . By construction, every index of  $\mathbf{W}$  is then paired in  $K$ . But if a word has a pairing in which every index is paired, it follows from (A.13) that  $\lambda(\mathbf{W}) = 0$ , so that, by Lemma A.1.3,

$$W = [\mathbf{W}] = e. \quad (\text{A.62})$$

On the other hand, since  $J$  is reduced, every paired index in  $\mathcal{W}_-(\mathbf{V})$  is linked to a distinct index in  $\mathcal{W}_+(\mathbf{V}) \cup \mathcal{W}_0(\mathbf{V})$  (recall that linked indices necessarily belong to inverse letters). This implies that  $\mathcal{W}_0(\mathbf{V})$  has  $3(m - q)$  paired indices, with  $(m - q)$  indices corresponding to  $A$ ,  $B$  and  $C$  each. Since, in passing to  $\mathbf{W}$ , no indices in  $\mathcal{C}(\mathbf{V})$  were removed, it follows that  $W$  is of the form

$$W = f_0 A^{m-q} B^{m-q} C^{m-q} f_0^{-1} \prod_{j=1}^m f_j (CBA)^{-1} f_j^{-1} \prod_{k=1}^q g_k CBA g_k^{-1} = e, \quad (\text{A.63})$$

where  $f_j, g_k \in F(A, B, C)$ , and we have incorporated (A.62).

Let  $\Phi$  be the homomorphism from  $F(A, B, C)$  to  $F(A, B)$  given by

$$\Phi(A) = A, \quad \Phi(B) = B, \quad \Phi(C) = A^{-1}B^{-1}, \quad (\text{A.64})$$

Then  $\Phi(CBA) = \Phi((CBA)^{-1}) = e$ . Applying  $\Phi$  to (A.63) and conjugating by  $\Phi(f_0)^{-1}$ , we obtain

$$A^{m-q} B^{m-q} (A^{-1}B^{-1})^{m-q} = e. \quad (\text{A.65})$$

This implies  $m = q$ , in contradiction to (A.61).  $\square$

*Proof of Proposition 2.4.2.* Let  $\mathcal{Q}_{\mathbf{n}, \mathbf{p}}$  denote the set of words

$$\begin{aligned} \mathcal{Q}_{\mathbf{n}, \mathbf{p}} = \{ & (\mathbf{h}_0, A^i, B^j, C^k, \mathbf{h}_0^{-1}, \\ & \mathbf{h}_1, C^{-1}, B^{-1}, A^{-1}, \mathbf{h}_1^{-1}, \dots, \mathbf{h}_n, C^{-1}, B^{-1}, A^{-1}, \mathbf{h}_n^{-1}, \\ & \mathbf{h}_{n+1}, A, B, C, \mathbf{h}_{n+1}^{-1}, \dots, \mathbf{h}_{n+p}, A, B, C, \mathbf{h}_{n+p}^{-1}), \quad \mathbf{h}_t \in \mathcal{L}_3 \}. \end{aligned} \quad (\text{A.66})$$

Then  $\mathbf{U} \in \mathcal{Q}_{\mathbf{n}, \mathbf{p}}$  implies that  $U \in \mathcal{Q}_{n,p}$ . For  $\mathbf{U} \in \mathcal{Q}_{\mathbf{n}, \mathbf{p}}$ , let

$$E_{n,p}(\mathbf{U}) = \lambda(\mathbf{U}) - (i + j + k - (n + p) - 2). \quad (\text{A.67})$$

Then Proposition 2.4.2 is equivalent to showing that

$$E_{n,p}(\mathbf{U}) \geq 0. \quad (\text{A.68})$$

Following the proof of Proposition 2.4.1, we construct  $X \in \mathcal{Q}_{m,q}$  with  $m > 2q + 1$  such that

$$X = f_0 A^{m-q} B^{m-q} C^{m-q} f_0^{-1} \prod_{j=1}^m f_j (ABC)^{-1} f_j^{-1} \prod_{k=1}^q g_k ABC g_k^{-1} = e. \quad (\text{A.69})$$

Apply the homomorphism  $\Psi : F(A, B, C) \rightarrow F(A, B)$  defined by  $\Psi(A) = A$ ,  $\Psi(B) = B$ ,  $\Psi(ABC) = e$  to get that

$$A^{m-q}B^{m-q}(B^{-1}A^{-1})^{m-q} = e. \quad (\text{A.70})$$

This implies either  $m = q$  or  $m = q + 1$ , both of which contradict  $m > 2q + 1$  (as  $q \geq 0$ ).  $\square$

It remains to give the proof of Lemma A.3.1.

*Proof of Lemma A.3.1.* From Proposition A.2.2, we have that

$$D_{n,p}(\mathbf{U}) = 4(n + p) - [I]_W. \quad (\text{A.71})$$

Suppose that  $\mathcal{W}_-(\mathbf{U})$  contains indices which are unpaired in  $I$  (the argument for the case where  $\mathcal{W}_+(\mathbf{U})$  has unpaired indices is similar). Then at least one of the negative triples in  $\mathbf{U}$  contains letters with unpaired indices. Let  $t, t+1, t+2$  denote the (consecutive) indices of one such negative triple. Let  $v$  denote the number of these indices that are paired in  $I$ , so that, by assumption,  $0 \leq v \leq 2$ . Denote these paired indices by  $t + \alpha_u$ , where  $1 \leq u \leq v$  and  $\alpha_u = 0, 1$  or  $2$ . Let  $j_u$  denote the index to which  $t + \alpha_u$  is linked. Since  $I$  is reduced, we have that

$$j_u \in \mathcal{W}(\mathbf{U}). \quad (\text{A.72})$$

Let  $c_{u,1}, \dots, c_{u,m_u}$  denote the linking sequence from  $j_u$  to  $t + \alpha_u$ , so that

$$\{j_u, c_{u,1}\}, \{\bar{c}_{u,1}, c_{u,2}\}, \dots, \{\bar{c}_{u,m_u-1}, c_{u,m_u}\}, \{\bar{c}_{u,m_u}, t + \alpha_u\} \in I \quad (\text{A.73})$$

(of course, if  $v = 0$ , there are no such linking sequences). Let

$$R = \{t, t+1, t+2, \} \cup \{\cup_{u=1}^v \{c_{u,1}, \bar{c}_{u,1}, \dots, c_{u,m_u}, \bar{c}_{u,m_u}\}\}. \quad (\text{A.74})$$

Let us remove  $R$  from  $\mathbf{U}$  and  $I$  to get  $\mathbf{V}$  with pairing  $J$ . Since the indices in  $R \cap \mathcal{C}(\mathbf{U})$  occur in conjugate pairs and  $\mathbf{V}$  has one less negative triple than does  $\mathbf{U}$ , it follows that  $\mathbf{V} \in \mathcal{P}_{\mathbf{n}-1, \mathbf{p}}$ .

We argue that every index in  $\mathcal{C}(\mathbf{V})$  is paired in  $J$ , ie

$$[J]_C = |\mathcal{C}(\mathbf{V})| \quad (\text{A.75})$$

(note that  $J$  need not be optimal and we have not established that  $J$  is reduced, so this assertion does not follow from Proposition A.2.2). Take  $d \in \mathcal{C}(\mathbf{V})$ , and let  $c = \psi_R^{-1}(d)$ . Then

$$c \in \mathcal{C}(\mathbf{U}) - R. \quad (\text{A.76})$$

Since  $I$  is optimal and reduced, it follows from Proposition A.2.2 that  $c$  is paired to some index  $b$  in  $I$ . We claim that  $b \notin R$ . Given that this is so, it follows from (A.49) that  $d$  is indeed paired in  $J$ , to  $\psi_R(b)$  in fact.

To show that  $b \notin R$ , let us assume the contrary; then we have that  $b \in R$  is paired to  $c \notin R$ . Examination of (A.74) and (A.73) shows that the only indices in  $R$  that are paired to indices not in  $R$  are the  $c_{u,1}$ 's, which are paired to the  $j_u$ 's. It follows that  $b = c_{u,1}$  and  $c = j_u$  for some  $1 \leq u \leq v$ . But this would imply that  $j_u = c \in \mathcal{C}(\mathbf{U})$ , in contradiction to (A.72).

Since  $J$  need not be an optimal pairing for  $\mathbf{V}$ , we have the inequality (rather than equality)

$$\begin{aligned}\lambda(\mathbf{V}) &\leq L(\mathbf{V}) - 2|J| = |\mathcal{W}(\mathbf{V})| + |\mathcal{C}(\mathbf{V})| - [J]_W - [J]_C \\ &= |\mathcal{W}(\mathbf{V})| - [J]_W = i + j + k + 3(n - 1 + p) - [J]_W, \quad (\text{A.77})\end{aligned}$$

where we have used (A.75) in the second equality. It follows that

$$D_{n-1,p}(\mathbf{V}) \leq 4(n - 1 + p) - [J]_W. \quad (\text{A.78})$$

From (A.49), the only paired indices in  $\mathcal{W}(\mathbf{U})$  which are not mapped by  $\psi_R$  into paired indices in  $\mathcal{W}(\mathbf{V})$  are those which belong to  $R$  itself and those which are paired with indices in  $R$ . There are  $v$  of the former – namely the  $t + \alpha_u$ 's – and  $v$  of the latter – namely the  $j_u$ 's, where  $1 \leq u \leq v$ . Therefore,

$$[J]_W = [I]_W - 2v. \quad (\text{A.79})$$

From (A.71), (A.78) and (A.79), we conclude that

$$D_{n-1,p}(\mathbf{V}) \leq D_{n,p}(\mathbf{U}) - 2(2 - v) < 0, \quad (\text{A.80})$$

as  $v \leq 2$  and  $D_{n,p}(\mathbf{U}) < 0$ , by assumption.  $\square$

## References

- [1] R. ADAMS, 1975 Sobolev Spaces *Academic Press*.
- [2] H. BREZIS, J.M. CORON, E.H. LIEB, 1986 Harmonic maps with defects. *Communications in Mathematical Physics* **107**, 649-705.
- [3] R. BROWN, H. SCHIRMER, Nielsen root theory and Hopf degree theory, *Pacific J. Math.* 198 (2001) 4980.
- [4] P. G. DE GENNES, 1974 The physics of liquid crystals. *Oxford, Clarendon Press*.
- [5] F. DUZAAR & K. STEFFEN, 1989 A partial regularity theorem for harmonic maps at a free boundary. *Asymptotic Analysis* **2**, 299-343.
- [6] J.EELLS & B.FUGLEDE, 2001 Harmonic Maps Between Riemannian Polyhedra. *Cambridge University Press*.
- [7] S. KITSON & A. GEISOW, 2002 Controllable alignment of nematic liquid crystals around microscopic posts: Stabilization of multiple states. *Applied Physics Letters* **80**, 3635 – 3637.
- [8] F. H. LIN & C. LIU, 2001 Static and Dynamic Theories of Liquid Crystals. *Journal of Partial Differential Equations* **14** no. 4, 289-330.
- [9] W. MAGNUS, A. KARRAS & D. SOLITAR, 1976 Combinatorial group theory. *Dover*.

- [10] A. MAJUMDAR, J. M. ROBBINS & M. ZYSKIN, 2004 Lower Bound for Energies of Harmonic Tangent Unit-Vector Fields on Convex Polyhedra. *Lett. Math. Phys.* **70**, 169 – 183.
- [11] A. MAJUMDAR, J. M. ROBBINS & M. ZYSKIN, 2004 Elastic energy of liquid crystals in convex polyhedra. *Journal of Physics A - Mathematics and General* **37**, L573–L580.
- [12] A. MAJUMDAR, J. M. ROBBINS & M. ZYSKIN, 2006 Elastic energy for reflection-symmetric topologies. *Journal of Physics A* **39**, 2673 – 2687.
- [13] A. MAJUMDAR, 2006 Liquid crystals and tangent unit-vector fields on polyhedral geometries, Ph.D. thesis.
- [14] A. MAJUMDAR, J. M. ROBBINS & M. ZYSKIN, 2006 Energies of  $S^2$ -valued harmonic maps on polyhedra with tangent boundary conditions. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*. **In press**.
- [15] A. MAJUMDAR, C. J. P. NEWTON, J. M. ROBBINS & M. ZYSKIN, 2007 Topology and bistability in liquid crystal devices. *Phys. Rev. E* **75**, 051703–051714.
- [16] A. MAJUMDAR, J. M. ROBBINS & M. ZYSKIN, 2008 Liquid crystals and harmonic maps in polyhedral domains. in *Analysis and Stochastics of Growth Processes and Interface Models*, eds Peter Morters, Roger Moser, Mathew Penrose, Hartmut Schwetlick, and Johannes Zimmer. Oxford University Press, pp 306 - 326
- [17] TIM RILEY; private communication.
- [18] J. M. ROBBINS & M. ZYSKIN, 2004 Classification of unit-vector fields in convex polyhedra with tangent boundary conditions. *Journal of Physics A* **37**, 10609-10623.
- [19] M, SPIVAK, 1990 *A Comprehensive Introduction to Differential Geometry*, Vol. 2, 2nd ed. *Berkeley, CA: Publish or Perish Press*.
- [20] IAIN W. STEWART, 2004 *The Static and Dynamic Continuum Theory of Liquid Crystals*. *London, Taylor and Francis*.
- [21] E.G. VIRGA, 1994 *Variational Theories for Liquid Crystals*. *London, Chapman and Hall*.